

MOSTOW RIGIDITY AND THE PROPORTIONALITY PRINCIPLE FOR SIMPLICIAL VOLUME

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CONTENTS

1. Introduction	2
2. Hyperbolic Geometry	3
2.1. Models of Hyperbolic Space	3
2.2. Isometries and Geodesics	5
2.3. Maximal Volume Simplices	6
3. The Proportionality Principle	14
3.1. The Gromov Norm	14
3.2. Proportionality Principle for Hyperbolic Manifolds	15
3.3. Measure Homology and the General Proportionality Principle	20
4. Mostow Rigidity	24
4.1. Obtain and Lift a Homotopy Equivalence	24
4.2. Extend to $\partial\mathbb{H}^n$	25
4.3. Rigidity	28
4.4. Further Research Directions	30
References	31

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1. INTRODUCTION

Many deep theorems in mathematics live at the interface between geometry and topology. Topology, roughly, is the study of shapes and spaces up to squishing and stretching. For instance, topology can tell us the essential differences between an infinite plane, a sphere, and a torus. But to repeat an old joke, a topologist is someone who cannot tell the difference between a doughnut and a coffee cup. The cup part of a coffee cup is topologically trivial; one could squish down the sides of the cup into the base, and then squish the base down to a point, leaving only the handle which is identical to a doughnut or solid torus. Geometry, on the other hand, concerns itself with the messy details of measuring lengths and angles, volumes and curvatures.

A principal object in topology is a *topological manifold*, a space which locally looks like the familiar Euclidean space, but these local patches may globally connect up in complicated ways. For example, a sphere is a simple 2-dimensional manifold; it locally looks like a plane if you zoom in enough, but if you travel in any fixed direction for long enough you will eventually return to your starting location. To do geometry, one needs some additional structure. The central object of study in geometry is a *Riemannian manifold*, a topological manifold equipped with a *metric*, which is essentially a prescription for how to measure infinitesimal lengths and angles at each point. The stretching and squishing which is ignored in topology manifests in geometry as deformations of the metric. One might reasonably expect that geometric and topological information exist at fundamentally different levels, that geometric methods would yield information only about the particular metric and not about the underlying space, and conversely that topological methods can't see the metric at all and thus can't say anything about it. In fact, the two are deeply intertwined.

For an example of this, consider the case of closed, connected, oriented 2-manifolds (i.e. surfaces). A topological investigation shows that there are infinitely many such surfaces, fully classified by a single nonnegative integer g , called the *genus*, which colloquially is the number of "holes". The genus 0 surface is the sphere; the genus 1 surface is the torus; the genus 2 surface may be obtained by cutting small patches off of two tori and gluing them together along the cut edges; etc. It is natural to ask what sorts of geometric behavior may arise when putting a metric on these various surfaces. In particular, we are interested in *homogeneous* metrics, where the geometry looks the same at every point. The key insight comes from the famous Gauss-Bonnet theorem.

Theorem 1.1 (Gauss-Bonnet). *Let M be a closed Riemannian 2-manifold with Gaussian curvature K . Then*

$$\int_M K \, dA = 2\pi\chi(M).$$

The left side of the equation is purely geometric; K stands for the *Gaussian curvature*, and the integral computes the total curvature of the surface. The right hand side is purely topological; it is simply a constant times $\chi(M)$, the *Euler characteristic* of M . The Euler characteristic may be computed by taking a triangulation of the surface and counting the number of faces in the triangulation, minus the number of edges, plus the number of vertices. The triangulation may be stretched and squished freely, so this is a topological invariant. In fact, the Euler characteristic is related to the genus by the simple formula

$$\chi(M) = 2 - 2g.$$

In particular, a phase transition occurs at $g = 1$. The sphere has positive Euler characteristic, the torus has 0 Euler characteristic, and all other closed orientable surfaces have negative Euler characteristic. By the Gauss-Bonnet theorem, it follows that a homogeneous metric on the sphere must have positive curvature, a homogeneous metric on the torus has no curvature, and a homogeneous metric on a surface of genus $g \geq 2$ has negative curvature. It is not hard to explicitly construct such metrics, and by scaling we may make the curvature precisely $+1$, 0 , and -1 in the three cases respectively. We say that the sphere has *spherical* or *elliptical* geometry, the torus has *Euclidean* or *flat* geometry, and the higher genus surfaces have *hyperbolic* geometry. More technically, these terms refer to the surfaces being locally isometric to a *model geometry*, either the sphere S^2 , Euclidean 2-space \mathbb{E}^2 , or hyperbolic 2-space \mathbb{H}^2 . These three geometric models cover all of the cases, and the topology uniquely determines which of the three geometries is relevant. This result is known as the uniformization theorem.

The situation is more complicated in dimension 3, but topology and geometry are just as tightly intertwined. The 3-manifold analogue of the uniformization theorem is the geometrization conjecture, more appropriately called the geometrization theorem since Perelman outlined a proof in a series of papers in 2002-03 [Per02][Per03b][Per03a]. The theorem states that every closed 3-manifold can be canonically cut into pieces along embedded spheres and tori such that each piece admits a finite volume geometric structure, described by one of eight different model geometries. These model geometries include the 3-dimensional analogues of the 2-dimensional models, S^3 , \mathbb{E}^3 , and \mathbb{H}^3 .

The uniformization and geometrization theorems are deep and rightfully celebrated, but there is little hope for such a clean characterization in dimensions 4 or greater. Even in dimensions 2 and 3, these theorems leave open many lines of inquiry. Given a manifold, one might ask not only what types of geometries it can support, but also how unique such a metric is, and how much of the geometric information may be extracted from the underlying topology.

In this thesis we explore the particular case of hyperbolic manifolds. Chapter 2 is devoted to reviewing the basic properties of hyperbolic space. In section 2.3, we prove the existence of maximal volume hyperbolic simplices, an unintuitive property which underlies much of the peculiar behavior of hyperbolic manifolds. In chapter 3 we introduce topological invariants, the Gromov norm and simplicial volume, and show that they capture geometric data. The main result of the chapter is the proportionality principle, which states that simplicial volume is directly proportional to the actual geometric volume. Finally, in chapter 4 we present Gromov's proof of the Mostow rigidity theorem, which states that if a manifold in dimension at least 3 admits a hyperbolic metric, then that metric is unique. This means that for hyperbolic manifolds, the geometry is entirely determined by the topology. Determining specifically which topological invariants of hyperbolic manifolds correspond to particular geometric quantities is an active area of research.

2. HYPERBOLIC GEOMETRY

We begin with a brief review of the basics of hyperbolic geometry. No background knowledge about hyperbolic space is explicitly assumed, and the properties used in the later chapters are either proved or at least explained here, but the treatment is necessarily a bit rushed. The reader is encouraged to consult Ratcliffe [Rat06] for a thorough treatment.

2.1. Models of Hyperbolic Space.

We will find it useful at various points to work in three different models of hyperbolic n -space. A model consists of a subset of \mathbb{R}^n and a Riemannian metric on that subset. Each model will be given its own notation, and the more general symbol \mathbb{H}^n should be understood to refer to any one of them when particular models are not needed.

Definition 2.1.

The *Poincaré ball model*:

$$B^n = \{x \in \mathbb{R}^n : |x| < 1\}$$

$$ds_B^2 = \frac{4|dx|^2}{(1 - |x|^2)^2}$$

The *upper half-space model*, also due to Poincaré:

$$U^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$$

$$ds_U^2 = \frac{|dx|^2}{x_n^2}$$

The *Klein model*:

$$K^n = \{x \in \mathbb{R}^n : |x| < 1\}$$

$$ds_K^2 = \frac{|dx|^2}{1 - |x|^2} + \frac{\langle x, dx \rangle^2}{(1 - |x|^2)^2}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on \mathbb{R}^n .

The models are related by the maps $\mu : U^n \rightarrow B^n$ and $\nu : B^n \rightarrow K^n$ given by

$$\mu(x) = -e_n + \frac{2}{|x + e_n|^2}(x + e_n)$$

$$\nu(x) = \frac{2x}{1 + |x|^2}$$

where $e_n = (0, \dots, 0, 1)$. Note that $\mu = \mu^{-1}$. We will usually prefer to work in the Poincaré ball or upper half-space models, as these are conformally equivalent to the standard Euclidean metric $ds_E^2 = |dz|^2$. That is, they differ from the Euclidean metric at each point only by scaling, so angles measured in these metrics agree with the standard Euclidean angles. Angles in the Klein model may be distorted. The principal advantage of the Klein model is that its geodesics are straight lines, the same as the Euclidean geodesics, as we will see below.

The metric ds_B^2 blows up as x approaches the boundary sphere ∂B^n . Indeed, the distance from the origin to the boundary is given by the integral

$$\int_0^1 \frac{2 dx}{1 - x^2}$$

which diverges, so the boundary sphere is infinitely far away from the interior of hyperbolic space. It is commonly called the *sphere at infinity*. Points on the sphere at infinity are called *ideal* points, as opposed to points properly within \mathbb{H}^n which are called *finite*. The sphere at infinity in the Klein model is the same, $\partial K^n = \partial B^n = S^{n-1}$, and the map $\nu : B^n \rightarrow K^n$ extends to the identity map on ∂B^n . In the upper half-space model, the sphere at infinity is really the boundary of U^n in the 1-point compactification of \mathbb{R}^n . That is, it consists of the boundary plane $\partial U^n = \{x \in \mathbb{R}^n : x_n = 0\}$ along with the compactifying point ∞ .

2.2. Isometries and Geodesics.

We let $I(M)$ denote the isometry group of a Riemannian manifold M . The group $I(\mathbb{H}^n)$ should be understood as abstractly isomorphic to the concrete groups $I(U^n)$, $I(B^n)$, and $I(K^n)$.

The symmetries of the models allow us to infer a great deal about the isometries and geodesics of \mathbb{H}^n without extensive analysis. Consider first the upper half-space model. The conformal factor in the metric ds_U^2 depends only on the height x_n , so all Euclidean isometries acting on the first $n-1$ coordinates are isometries of U^n . In particular, horizontal translations are isometries. Furthermore, length in U^n is inversely proportional to height, so a dilation $m_c(x) = cx$ is an isometry, since it scales length and height equally.

$$m_c^* ds_U^2 = \frac{|c \cdot dx|^2}{(cx_n)^2} = \frac{|dx|^2}{x_n^2} = ds_U^2$$

Together, horizontal translations and dilations act transitively on U^n , so \mathbb{H}^n is homogeneous. Now switch to the Poincaré ball model. The metric ds_B^2 is spherically symmetric, so the isometries of B^n include the orthogonal group $O(n)$. In particular, B^n is isotropic at the origin. Since \mathbb{H}^n is homogeneous, this shows that \mathbb{H}^n is isotropic everywhere.

As for geodesics, symmetry immediately shows that vertical lines in U^n and diameters in B^n are geodesics. By repeatedly applying the map μ relating U^n and B^n , along with the isometries of U^n and B^n discussed above, we see that geodesics in U^n and B^n are circular arcs perpendicular to the boundary sphere. The vertical lines in U^n and diameters in B^n represent the special case where the circle contains ∞ . Applying the map $\nu : B^n \rightarrow K^n$ takes the circular geodesics in B^n to straight lines in K^n with the same ideal endpoints, so the geodesics in K^n are simply Euclidean chords of S^{n-1} . Since a geodesic is fully determined by a tangent vector at a single point, we know that these are all the geodesics of \mathbb{H}^n .

Importantly, a geodesic is uniquely defined by two ideal endpoints. If two geodesics share an endpoint, then they become arbitrarily close to one another as they go off to infinity. (This is clearest in U^n , when the shared endpoint is ∞ .) In this case we say they are *asymptotic* to one another, a property preserved under isometry. Using this, every isometry of \mathbb{H}^n can be uniquely extended to a continuous map of $\partial\mathbb{H}^n$ (continuous in the inherited topology from \mathbb{R}^n in any of the models). Notationally, we will use the same symbol for the extended map as for the original isometry. We note that $O(n)$ acts transitively on ∂B^n . Switching back to U^n , this implies that there are isometries of U^n which, extended to the sphere at infinity, take any given ideal point to the distinguished point ∞ . Horizontal translations and dilations of U^n then act transitively on pairs of points in the plane ∂U^n while fixing ∞ . Thus $I(\mathbb{H}^n)$ acts transitively on ordered triplets of ideal points.

Lemma 2.2. *An isometry $\phi \in I(U^n)$ which fixes ∞ restricts to a Euclidean similarity on the plane ∂U^n .*

Proof. Consider a horizontal hypersurface $\Sigma_h = \{x \in U^n : x_n = h\}$ for fixed $h > 0$ in \mathbb{R} . (Note: This is not a geodesic hyperplane. In fact this is a *horosphere*, a spherical surface in U^n , the limit of spheres in the ds_U^2 metric whose centers are going off to ∞ . Horospheres “centered” at other ideal points in ∂U^n turn out to be Euclidean spheres tangent to the boundary plane.) Σ_h is perpendicular to all vertical geodesics. An isometry $\phi \in I(U^n)$ which fixes ∞ sends vertical geodesics to other vertical geodesics and must preserve that perpendicularity, so it must take Σ_h to another horizontal hypersurface $\Sigma_{h'}$.

Now consider three ideal points $p_1, p_2, p_3 \in \partial U^n$ which define a Euclidean triangle T in $\mathbb{R}^{n-1} = \partial U^n$, and the vertical geodesics starting at those points. The geodesics intersect

Σ_h at three points defining the Euclidean triangle T_h , which is simply T translated up by h . Then $\phi(T_h)$ is isometric to T_h in U^n and projects down to $\phi(T)$. The only difference between the metric restricted to Σ_h and that restricted to $\Sigma_{h'}$ is the scaling factor h/h' , so we conclude that T and $\phi(T)$ are similar triangles. Since this holds for any triplet of points and ϕ must either be globally orientation preserving or reversing, we conclude that ϕ acts on ∂U^n by a Euclidean similarity. \square

The action on the boundary dictates how ϕ acts on geodesics, which in turn uniquely defines ϕ on \mathbb{H}^n because a point may be specified as the intersection of two geodesics. The isometries of U^n discussed above account for all Euclidean similarities of ∂U^n . Additionally, we have seen that $O(n)$ acts transitively on ∂B^n in the Poincaré ball. Thus the lemma implies that the isometries simply identified above by the symmetry of the models do in fact generate $I(\mathbb{H}^n)$.

Remark. Hyperbolic geometry violates Euclid's parallel postulate. Given any hyperplane L and a point $p \notin L$, there are infinitely many hyperplanes through p which do not meet L .

For another curious property, drawing a triangle in B^2 quickly demonstrates that the sum of the internal angles of any (non-degenerate) triangle in \mathbb{H}^n is strictly less than π . In fact an ideal triangle, one whose vertices are all ideal points, has internal angles equal to 0.

It is worth briefly giving, without proofs, another description of the isometry group $I(\mathbb{H}^n)$.

Definition 2.3. *Inversion* through the sphere S of radius r centered at $x_0 \in \mathbb{R}^n$ is the map of the 1-point compactification $\mathbb{R}^n \cup \{\infty\}$ given by

$$\rho(x) = x_0 + \frac{r^2}{|x - x_0|^2}(x - x_0).$$

The map μ given above is one such inversion, with $x_0 = -e_n$ and $r = \sqrt{2}$. Reflections through hyperplanes may be considered degenerate cases of inversions, the hyperplane being a sphere containing ∞ . Inversions satisfy $\rho = \rho^{-1}$ and are orientation reversing. ρ exchanges the interior and exterior of the sphere (or the two sides of the hyperplane), and in particular the points x_0 and ∞ (reflection through a hyperplane preserves ∞). The fixed point locus of ρ is precisely the sphere S . One may check that inversions take circles to circles and spheres to spheres, where straight lines and hyperplanes are considered to be circles and spheres containing ∞ . Additionally inversions are conformal, i.e. they preserve angles. An inversion ρ preserves U^n (respectively B^n) if and only if S is perpendicular to ∂U^n (∂B^n). Finally, one may check that an inversion preserving U^n or B^n is an isometry of the hyperbolic metric in that model.

Definition 2.4. A *Möbius transformation* of $\mathbb{R}^n \cup \{\infty\}$ is a composition of inversions through spheres and reflections through hyperplanes.

Theorem 2.5 ([Rat06], Theorems 5.2.10, 5.2.11). *The groups of isometries $I(U^n)$ and $I(B^n)$ are equal to the groups of Möbius transformations fixing U^n and B^n respectively.*

2.3. Maximal Volume Simplices.

Definition 2.6. A *hyperbolic* or *Euclidean n -simplex* is the convex hull of $n + 1$ points in \mathbb{H}^n or \mathbb{E}^n respectively. (This is not to be confused with a *singular n -simplex*, which may be any map of the standard n -simplex into a topological space.) A simplex is said to be *regular* if any permutation of its vertices can be realized by an isometry.

We finish our basic treatment of hyperbolic geometry by proving the following theorem due to Haagerup and Munkholm [HM81]. Borrowing their notation, in this section $\tau[n]$ will denote an ideal hyperbolic n -simplex, $\tau_0[n]$ will denote a regular ideal hyperbolic n -simplex, $\sigma[n]$ will denote a Euclidean simplex inscribed in a sphere of radius 1, and $\sigma_0[n]$ will denote a regular such simplex. $\Delta[n]$ will denote an arbitrary hyperbolic simplex.

Theorem 2.7. *For $n \geq 2$, the set of possible volumes for a hyperbolic n -simplex $\Delta[n] \subset \mathbb{H}^n$ is bounded. Furthermore, a particular n -simplex Δ has volume*

$$\text{vol}(\Delta) = v_n = \sup_{\Delta[n] \subset \mathbb{H}^n} \text{vol}(\Delta[n])$$

if and only if Δ is ideal and regular.

Remark. This is a fairly technical result, but it is a key ingredient in the main results of the next two chapters. While the proportionality principle does not strictly depend on this, indeed we will see that it holds for non-hyperbolic manifolds as well, the finiteness of the constant v_n makes the proportionality principle meaningful in the hyperbolic case. Then the fact that the maximal volume v_n is achieved only by regular ideal simplices turns out to be a key point in proving Mostow rigidity.

We begin the proof with a simple observation.

Lemma 2.8. *Any simplex $\Delta \subset \mathbb{H}^n$ is properly contained within an ideal simplex.*

Proof. In the Poincaré ball model, we may translate Δ by an isometry so that $0 \in \Delta$. Now send each vertex p out to the ideal point $p/|p| \in \partial B^n$. It is clear that the resulting ideal simplex contains Δ . \square

By the lemma, it suffices to consider ideal simplices in proving Theorem 2.7. We proceed with the proof by induction with two base cases.

Proof of Theorem 2.7 for $n = 2$. As discussed in the previous section, the isometries of \mathbb{H}^n act transitively on all triplets of ideal points. Thus all ideal triangles are isometric in \mathbb{H}^2 ; the regularity statement is vacuous. For ease of computation, we may place the vertices at $(\pm 1, 0)$ and ∞ in U^n . The area of the ideal triangle is then

$$\begin{aligned} v_2 &= \int_{-1}^1 \int_{\sqrt{1-x_1^2}}^{\infty} \frac{dx_2 dx_1}{x_2^2} \\ &= \int_{-1}^1 \frac{dx_1}{\sqrt{1-x_1^2}} \\ &= \arcsin(x_1) \Big|_{-1}^1 \\ &= \pi \end{aligned} \quad \square$$

Proof of Theorem 2.7 for $n = 3$. Given an ideal simplex $\Delta = \Delta[3]$ with distinguished vertex p_0 , we may by isometries place $\Delta \subset U^3$ with p_0 at ∞ . The faces of Δ containing p_0 are then vertical planes. The dihedral angles between those faces can be measured by taking a horizontal slice $\Delta \cap \{x \in U^3 : x_3 = h\}$ for fixed $h > 0$ in \mathbb{R} . Assuming h is taken sufficiently large to avoid the face of Δ opposite from p_0 , this slice is independent of h , and is a Euclidean triangle. Thus the sum of the dihedral angles at each vertex of an ideal tetrahedron must be exactly π .

Now label the remaining vertices of Δ as p_1, p_2, p_3 , and let θ_{ij} denote the dihedral angle at the edge connecting p_i and p_j . Then the angle sum conditions at the various vertices give

$$\begin{aligned} 2\theta_{01} &= (\pi - \theta_{02} - \theta_{03}) + (\pi - \theta_{12} - \theta_{13}) \\ &= (\pi - \theta_{02} - \theta_{12}) + (\pi - \theta_{03} - \theta_{13}) \\ &= 2\theta_{23} \end{aligned}$$

By rearranging indices, we conclude that the dihedral angles at opposite edges are equal. The moduli space of ideal tetrahedra is therefore parameterized by the three dihedral angles incident to a single vertex, which we now label α, β , and γ , subject to the constraint that they must be nonnegative and sum to π . Lobachevsky provided the following explicit formula for the volume of the simplex in terms of those angles ([Thu02], Theorem 7.2.1).

$$\text{vol}(\Delta) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma),$$

where

$$\Lambda(\theta) = - \int_0^\theta \log(2 \sin t) dt$$

is known as the *Lobachevsky function*. Using this formula, we may solve for the maximal volume simplex using Lagrange multipliers.

$$\begin{aligned} \text{grad}(\alpha + \beta + \gamma) &= \text{grad}(\text{vol}(\Delta)) \\ \frac{\partial \text{vol}(\Delta)}{\partial \alpha} &= \frac{\partial \text{vol}(\Delta)}{\partial \beta} = \frac{\partial \text{vol}(\Delta)}{\partial \gamma} \\ -\log(2 \sin \alpha) &= -\log(2 \sin \beta) = -\log(2 \sin \gamma) \\ \sin \alpha &= \sin \beta = \sin \gamma \end{aligned}$$

The above, together with the fact that $\alpha + \beta + \gamma = \pi$, implies that $\alpha = \beta = \gamma = \pi/3$ so Δ is regular. This gives the unique critical point of the volume functional within the allowed parameter space. As the boundary of the parameter space consists of degenerate tetrahedra, with one dihedral angle 0 and thus volume 0, this critical point must be the global maximum. Plugging in, we have

$$v_3 = -3 \int_0^{\pi/3} \log(2 \sin t) dt \approx 1.01494. \quad \square$$

Now we prepare for the induction step with two important lemmas.

Lemma 2.9.

$$\frac{n-1}{n^2} < \frac{\text{vol}(\tau_0[n+1])}{\text{vol}(\tau_0[n])} < \frac{1}{n}.$$

Proof. Define the function

$$\Phi_n(\alpha) = \int_{\sigma_0[n]} \frac{dV}{(1-|x|^2)^\alpha}$$

where dV is the Euclidean volume element.

First, we place $\tau_0[n]$ in the upper half-space model with one vertex at ∞ . By dilating and translating, we may assume that the other n vertices p_1, \dots, p_n lie on the unit sphere $S^{n-2} \subset \partial U^n$. Since $\tau_0[n]$ is regular, there exist isometries in $I(U^n)$ realizing every permutation of the p_i while fixing ∞ . By Lemma 2.2, the restrictions of these isometries are Euclidean similarities of ∂U^n , but they also must preserve the unit sphere so in fact they are Euclidean

isometries. Consequently the vertical projection of $\tau_0[n]$ onto ∂U^n is a regular Euclidean simplex.

The bottom face of $\tau_0[n]$ is part of the unit hemisphere in U^n . The other faces are all vertical planes. The volume is given by the following integral.

$$\begin{aligned} \text{vol}(\tau_0[n]) &= \int_{\sigma_0[n-1]} \int_{\sqrt{1-r^2}}^{\infty} \frac{dx_n}{x_n^n} dV \\ &= \frac{1}{n-1} \int_{\sigma_0[n-1]} \frac{dV}{(1-r^2)^{(n-1)/2}} \\ (n-1)\text{vol}(\tau_0[n]) &= \Phi_{n-1} \left(\frac{n-1}{2} \right) \end{aligned} \quad (2.1)$$

$$n \text{vol}(\tau_0[n+1]) = \Phi_n \left(\frac{n}{2} \right) \quad (2.2)$$

Alternatively, we can place $\tau_0[n]$ in the Klein model, where it is precisely a regular Euclidean n -simplex. To compute the volume there, we need the volume element in the Klein model. We may expand the metric ds_K^2 introduced in Definition 2.1 into the matrix for the metric tensor g_K , with entries

$$(g_K)_{ij} = \frac{x_i x_j}{(1-|x|^2)^2} + \frac{\delta_{ij}}{1-|x|^2}.$$

By spherical symmetry the volume element at $x \in K^n$ is the same as that at $|x|e_1$. At that point, the metric tensor has entries

$$(g_K)_{ij}(|x|e_1) = \begin{cases} \frac{1}{(1-|x|^2)^2} & i = j = 1 \\ \frac{1}{1-|x|^2} & i = j \neq 1 \\ 0 & i \neq j \end{cases}$$

The volume element is the square root of the determinant of the metric tensor times the Euclidean volume element.

$$dV_K(x) = \frac{dV}{(1-|x|^2)^{(n+1)/2}}$$

The volume of $\tau_0[n]$ is then

$$\text{vol}(\tau_0[n]) = \int_{\sigma_0[n]} \frac{dV}{(1-|x|^2)^{(n+1)/2}} = \Phi_n \left(\frac{n+1}{2} \right). \quad (2.3)$$

Now consider the vector field

$$X_x = \frac{x}{(1-|x|^2)^{(n-1)/2}}$$

defined on the unit ball. The divergence theorem applied to X and $\sigma_0[n]$ gives

$$\int_{\sigma_0[n]} \text{div}(X) dV = \int_{\partial\sigma_0[n]} \langle X, \hat{n} \rangle dV, \quad (2.4)$$

where \hat{n} is the outward unit normal vector. On the left side, we compute

$$\begin{aligned} \frac{\partial}{\partial x_i} \left(\frac{x_i}{(1 - |x|^2)^{(n-1)/2}} \right) &= \frac{1}{(1 - |x|^2)^{(n-1)/2}} + (n-1) \frac{x_i^2}{(1 - |x|^2)^{(n+1)/2}} \\ \operatorname{div}(X) &= \frac{n}{(1 - |x|^2)^{(n-1)/2}} + \frac{(n-1)|x|^2}{(1 - |x|^2)^{(n+1)/2}} \\ &= \frac{1}{(1 - |x|^2)^{(n-1)/2}} + \frac{n-1}{(1 - |x|^2)^{(n+1)/2}} \\ \int_{\sigma_0[n]} \operatorname{div}(X) dV &= \Phi_n \left(\frac{n-1}{2} \right) + (n-1) \Phi_n \left(\frac{n+1}{2} \right) \end{aligned}$$

On the right side of (2.4), we note by symmetry that each face of $\sigma_0[n]$ contributes the same amount. If we label the vertices of $\sigma_0[n]$ as p_0, \dots, p_n , then a point in the simplex can be written in barycentric coordinates as

$$x = \sum_i t_i p_i,$$

where $t_i \geq 0$ and $\sum_i t_i = 1$. The barycenter itself is

$$0 = \frac{1}{n+1} \sum_i p_i,$$

and the i -th face is the locus defined by $t_i = 0$. At $x \in \partial_j \sigma_0[n]$, the normal vector is $\hat{n} = -p_j$, and the inner product $\langle x, \hat{n} \rangle$ gives the distance a from that face to the origin, which is thus independent of the specific point x . Let

$$x = b_j = \frac{1}{n} \sum_{i \neq j} p_i$$

be the barycenter of the j -th face. Then we have

$$\begin{aligned} \langle p_j, \frac{1}{n} \sum_i p_i \rangle &= \langle p_j, \frac{1}{n} p_j + x \rangle \\ \langle p_j, 0 \rangle &= \frac{1}{n} + \langle p_j, x \rangle \\ a = \langle \hat{n}, x \rangle &= \frac{1}{n} \end{aligned}$$

Let c be the distance from b_j to a vertex p_i , $i \neq j$, and let $y = x - b_j$. By two applications of the Pythagorean theorem,

$$1 - |x|^2 = 1 - (a^2 + |y|^2) = c^2 - |y|^2.$$

The right hand side of (2.4) thus becomes

$$\int_{\partial \sigma_0[n]} \langle X, \hat{n} \rangle dV = \frac{n+1}{n} \int_{\partial_j \sigma_0[n]} \frac{dV}{(c^2 - |y|^2)^{(n-1)/2}}.$$

We then scale the variable y by c^{-1} so that the integral is taken over $\sigma_0[n-1]$. This requires the replacement $dV \mapsto c^{n-1}dV$.

$$\begin{aligned} \int_{\partial\sigma_0[n]} \langle X, \hat{n} \rangle dV &= \frac{n+1}{n} \int_{\sigma_0[n-1]} \frac{c^{n-1} dV}{(c^2 - c^2|y|^2)^{(n-1)/2}} \\ &= \frac{n+1}{n} \Phi_{n-1} \left(\frac{n-1}{2} \right) \end{aligned}$$

We plug back in to (2.4) and apply the previous equations (2.1) and (2.3).

$$\begin{aligned} \Phi_n \left(\frac{n-1}{2} \right) + (n-1)\Phi_n \left(\frac{n+1}{2} \right) &= \frac{n+1}{n} \Phi_{n-1} \left(\frac{n-1}{2} \right) \\ \Phi_n \left(\frac{n-1}{2} \right) + (n-1)\text{vol}(\tau_0[n]) &= \frac{n+1}{n} (n-1)\text{vol}(\tau_0[n]) \\ \Phi_n \left(\frac{n-1}{2} \right) &= \frac{n-1}{n} \text{vol}(\tau_0[n]) \end{aligned} \tag{2.5}$$

Finally, compare equations (2.5), (2.2), and (2.3). $\Phi_n(\alpha)$ is a monotone increasing function of α , so these imply the desired inequalities.

$$\begin{aligned} \frac{n-1}{n} \text{vol}(\tau_0[n]) &< n \text{vol}(\tau_0[n+1]) < \text{vol}(\tau_0[n]) \\ \frac{n-1}{n^2} &< \frac{\text{vol}(\tau_0[n+1])}{\text{vol}(\tau_0[n])} < \frac{1}{n} \end{aligned} \quad \square$$

Together with the base cases given above, this lemma has shown that $\tau_0[n]$ has finite volume for all $n \geq 2$. The next lemma will help us compare regular simplices to non-regular simplices.

Lemma 2.10. *Let $f : (0, 1] \rightarrow \mathbb{R}$ be a strictly concave continuous function and $\sigma[n]$ be an arbitrary Euclidean simplex inscribed in the unit sphere with barycenter b . Then*

$$\frac{1}{\text{vol}(\sigma[n])} \int_{\sigma[n]} f(1 - |x|^2) dV \leq \frac{1}{\text{vol}(\sigma_0[n])} \int_{\sigma_0[n]} f((1 - |b|^2)(1 - |x|^2)) dV,$$

and equality holds if and only if $\sigma[n]$ is regular.

Proof. As in the previous lemma, label the vertices of $\sigma[n]$ as p_0, \dots, p_n and pass to barycentric coordinates. The left hand integral then becomes an integral over the standard n -simplex

$$\Delta[n] = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} : t_i \geq 0, \sum_i t_i = 1\}.$$

$$\frac{1}{\text{vol}(\sigma[n])} \int_{\sigma[n]} f(1 - |x|^2) dV = \int_{\Delta[n]} f \left(1 - \left| \sum_i t_i p_i \right|^2 \right) d\mu$$

where μ is the standard Lebesgue measure on $\Delta[n]$ normalized to have total measure 1. The measure μ is invariant under permutations of the coordinates, so we can take an expected

value over all such permutations.

$$\begin{aligned} \frac{1}{\text{vol}(\sigma[n])} \int_{\sigma[n]} f(1 - |x|^2) dV &= \mathop{E}_{\pi \in S_n} \left(\int_{\Delta[n]} f \left(1 - \left| \sum_i t_{\pi(i)} p_i \right|^2 \right) d\mu \right) \\ &\leq \int_{\Delta[n]} f \left(1 - \mathop{E}_{\pi \in S_n} \left(\left| \sum_i t_{\pi(i)} p_i \right|^2 \right) \right) d\mu \end{aligned}$$

by Jensen's inequality. Skipping through some of the algebraic simplifications, we compute

$$\begin{aligned} \mathop{E}_{\pi \in S_n} \left(\left| \sum_i t_{\pi(i)} p_i \right|^2 \right) &= \sum_i t_i^2 + \mathop{E}_{\pi \in S_n} \left(\sum_{i \neq j} t_{\pi(i)} t_{\pi(j)} \langle p_i, p_j \rangle \right) \\ \mathop{E}_{\pi \in S_n} \left(\sum_{i \neq j} t_{\pi(i)} t_{\pi(j)} \right) &= \frac{1}{n(n+1)} \sum_{i \neq j} t_i t_j \\ &= \frac{1}{n(n+1)} \left(1 - \sum_i t_i^2 \right) \\ \sum_{i \neq j} \langle p_i, p_j \rangle &= \left| \sum_i p_i \right|^2 - \sum_i |p_i|^2 \\ &= (n+1)^2 |b|^2 - (n+1) \\ \frac{1}{\text{vol}(\sigma[n])} \int_{\sigma[n]} f(1 - |x|^2) dV &\leq \int_{\Delta[n]} f \left(\frac{n+1}{n} \left(1 - \sum_i t_i^2 \right) (1 - |b|^2) \right) d\mu \quad (2.6) \end{aligned}$$

Equality holds in (2.6) when it holds in Jensen's inequality. Since f is strictly concave, this happens if and only if the random variable being averaged over is constant, i.e. if and only if

$$\left| \sum_i t_{\pi(i)} p_i \right|^2 = \left| \sum_i t_i p_i \right|^2$$

for all permutations π and at all points $t \in \Delta[n]$. This is certainly true if $\sigma[n]$ is regular. Conversely, if it is true then in particular

$$|p_i + p_j|^2 = |p_k + p_\ell|^2$$

for any indices with $i \neq j, k \neq \ell$. Expanding, this implies

$$\langle p_i, p_j \rangle = \langle p_k, p_\ell \rangle$$

and then

$$|p_i - p_j| = |p_k - p_\ell|.$$

All the edges have equal lengths, so $\sigma[n]$ must be regular. Equality holds in (2.6) if and only if $\sigma[n]$ is regular.

Now let $g(x) = f(x(1 - |b|^2))$. g is strictly concave, so the preceding argument applies to g and $\sigma_0[n]$ to give the equivalent of (2.6) but with equality rather than inequality.

$$\frac{1}{\text{vol}(\sigma_0[n])} \int_{\sigma_0[n]} f((1 - |x|^2)(1 - |b|^2)) dV = \int_{\Delta[n]} f\left(\frac{n+1}{n} \left(1 - \sum_i t_i^2\right) (1 - |b|^2)\right) d\mu$$

Substituting back in to (2.6) gives the desired result. \square

End of proof of Theorem 2.7. At long last we can prove the inductive step. Having shown that regular ideal simplices have finite volume, all that remains is to show that a regular ideal simplex has volume at least that of any other ideal simplex. Let $n \geq 3$ and suppose the result holds for hyperbolic n -simplices. Define the function

$$f(t) = t^{-n/2} - n \frac{\text{vol}(\tau_0[n+1])}{\text{vol}(\tau_0[n])} t^{-(n+1)/2}$$

for $t \in (0, 1]$. The condition for f to be strictly concave is

$$f''(t) = \frac{n(n+2)}{4} t^{-(n+4)/2} - \frac{n(n+1)(n+3)}{4} \frac{\text{vol}(\tau_0[n+1])}{\text{vol}(\tau_0[n])} t^{-(n+5)/2} < 0$$

$$\frac{\text{vol}(\tau_0[n+1])}{\text{vol}(\tau_0[n])} > \frac{n+2}{(n+1)(n+3)} \sqrt{t}$$

By Lemma 2.9,

$$\frac{\text{vol}(\tau_0[n+1])}{\text{vol}(\tau_0[n])} \geq \frac{n-1}{n^2},$$

and in turn

$$\frac{n-1}{n^2} > \frac{n+2}{(n+1)(n+3)} \geq \frac{n+2}{(n+1)(n+3)} \sqrt{t}$$

for $n \geq 3$. Thus f is strictly concave.

Suppose $\tau[n+1] \subset U^{n+1}$ be an ideal simplex with one vertex at ∞ and the other vertices on the unit sphere in $\partial U^{n+1} = \mathbb{R}^n$. Let $\sigma[n]$ be the vertical projection of $\tau[n]$ down to ∂U^{n+1} , and let $\tau[n]$ be $\sigma[n]$ interpreted as an ideal simplex in the Klein model. Now we can apply Lemma 2.10 to $\sigma[n]$.

$$\frac{1}{\text{vol}(\sigma[n])} \int_{\sigma[n]} f(1 - |x|^2) dV \leq \frac{1}{\text{vol}(\sigma_0[n])} \int_{\sigma_0[n]} f((1 - |x|^2)(1 - |b|^2)) dV$$

Since the regular simplex $\sigma_0[n]$ is the largest Euclidean simplex which can be inscribed in the unit sphere, removing the volume factors only weakens the inequality. Then we expand out the definition of f and substitute using equations (2.2) and (2.3). Note from their derivations that these equations work for the non-regular simplices as we have defined them, in addition to the regular simplices.

$$n \text{vol}(\tau[n+1]) - n \frac{\text{vol}(\tau_0[n+1])}{\text{vol}(\tau_0[n])} \text{vol}(\tau[n]) \leq \frac{n \text{vol}(\tau_0[n+1])}{(1 - |b|^2)^{n/2}} - \frac{n \text{vol}(\tau_0[n+1])}{(1 - |b|^2)^{(n+1)/2}}$$

$$\leq 0$$

$$\frac{\text{vol}(\tau_0[n+1])}{\text{vol}(\tau[n+1])} \geq \frac{\text{vol}(\tau_0[n])}{\text{vol}(\tau[n])}$$

$$\geq 1 \quad (\text{by induction}) \quad \square$$

3. THE PROPORTIONALITY PRINCIPLE

This chapter will be devoted to introducing a topological analogue of volume and exploring its connection to the geometric volume. This connection will turn out to be particularly useful for hyperbolic manifolds, thanks to the existence of maximal volume hyperbolic simplices proven in the previous chapter. Our treatment at various points follows sections 11.4 and 11.5 of Ratcliffe [Rat06], section 6.1 of Thurston [Thu02], and notes by Butler [But12].

3.1. The Gromov Norm.

For a manifold M , let $C_k(M) = C_k(M; \mathbb{R})$ denote the vector space of singular k -chains on M with real coefficients, and $H_k(M) = H_k(M; \mathbb{R})$ the corresponding singular homology groups.

Definition 3.1. Let $\|\cdot\|_1 : C_k(M) \rightarrow \mathbb{R}$ be the ℓ^1 norm, defined by

$$\left\| \sum_i c_i \sigma_i \right\|_1 = \sum_i |c_i|.$$

Next, define the *Gromov norm* $\|\cdot\|$ on $H_k(M)$ by taking the infimum over all representatives of a homology class.

$$\|\alpha\| = \inf_{\substack{a \in C_k(M) \\ [a] = \alpha}} \|a\|_1$$

Despite the name, the Gromov norm is actually a seminorm, as nonzero homology classes may have norm 0. Finally, we define the *simplicial volume* $\|M\| = \|[M]\|$ as the Gromov norm of the fundamental class.

Remark. Morally, the simplicial volume may be thought of as the number of simplices required in the most efficient triangulation of M . The real coefficients confuse this intuition slightly, but we may think of them as correctly accounting for efficient triangulations by very large simplices which may cover M many times over. The hope is that with this flexibility, the infimum really can describe the most efficient triangulation, limited only by fundamental restrictions coming from the geometry of the manifold.

We begin by establishing the basic properties of the Gromov norm. Perhaps the most important property is that continuous maps cannot increase norm.

Lemma 3.2. *Let M and N be topological manifolds, $f : M \rightarrow N$ a map between them, and $\alpha \in H_k(M)$ a homology class. Then*

$$\|f_*(\alpha)\| \leq \|\alpha\|.$$

Proof. Suppose $a = \sum_i c_i \sigma_i$ is a singular chain with $[a] = \alpha$. Then $f_*(a) = \sum_i c_i f \circ \sigma_i$ is a chain representing $f_*(\alpha)$. It is possible that $f \circ \sigma_i = f \circ \sigma_j$ for $i \neq j$, so some cancelation is possible among the coefficients of $f_*(a)$. Consequently we have

$$\|f_*(\alpha)\| \leq \|f_*(a)\|_1 \leq \sum_i |c_i| = \|a\|_1.$$

Since this is true for all a representing α , we conclude

$$\|f_*(\alpha)\| \leq \|\alpha\|. \quad \square$$

Corollary 3.3. *Homotopy equivalences preserve the Gromov norm. In particular, the simplicial volume is a homotopy invariant.*

Proof. Suppose that $f : M \rightarrow N$ is a homotopy equivalence with homotopy inverse g . Then $(g \circ f)_* = g_* \circ f_*$ is the identity on homology. Let $\alpha \in H_k(M)$. By the lemma,

$$\|\alpha\| = \|g_* f_*(\alpha)\| \leq \|f_*(\alpha)\| \leq \|\alpha\|,$$

which forces

$$\|f_*(\alpha)\| = \|\alpha\|. \quad \square$$

Recall that a map $f : M \rightarrow N$ has *degree* d if $f_*([M]) = d[N]$. The lemma then implies

Corollary 3.4. *Suppose $f : M \rightarrow N$ is a continuous map with degree d . Then*

$$\|N\| \leq \frac{1}{d} \|M\|. \quad \square$$

In particular, if M admits a self-map of degree $d > 1$, then $\|M\| = 0$. This makes simplicial volume a useless invariant for a large class of manifolds.

Corollary 3.5. *Let M be a closed, connected, spherical or flat manifold. Then $\|M\| = 0$.*

Proof. Both the sphere S^n and the n -torus T^n admit self-maps of degree $d > 1$, so $\|S^n\| = \|T^n\| = 0$. A closed, connected, spherical manifold is covered by S^n , and a theorem of Bieberbach states that a closed, connected, flat manifold is covered by T^n . The covering map must have degree $d \geq 1$, so we must have $\|M\| = 0$. \square

Returning to the intuitive description of simplicial volume in Remark 3.1, this corollary reflects the fact that spherical and flat simplices can be arbitrarily large, wrapping around a spherical or toroidal manifold arbitrarily many times. In contrast, Theorem 2.7 suggests that simplicial volume will prove much more useful in the hyperbolic case.

3.2. Proportionality Principle for Hyperbolic Manifolds.

We say that M is a *hyperbolic manifold* if it is a topological manifold locally isometric to \mathbb{H}^n . Equivalently, M is hyperbolic if its universal cover is \mathbb{H}^n , in which case we have $M \cong \Gamma \backslash \mathbb{H}^n$ where Γ is a discrete group of isometries acting freely and properly discontinuously on \mathbb{H}^n .

Theorem 3.6 (Proportionality Principle for Hyperbolic Manifolds). *Let M be a closed, connected hyperbolic manifold. Then*

$$\text{vol}(M) = v_n \|M\|,$$

where v_n is the volume of a regular ideal hyperbolic n -simplex.

This theorem makes precise the intuition above; the most efficient triangulations of M would consist of simplices approaching the maximal simplices with volumes v_n . At least, that would be true if the triangulations consisted of the hyperbolic simplices discussed in the previous chapter, convex hulls of $n + 1$ points in \mathbb{H}^n . We must first show that we may restrict our attention from all singular simplices to this particular type.

Definition 3.7. Consider the standard n -simplex parameterized with barycentric coordinates,

$$\Delta[k] = \{(t_0, \dots, t_k) \in \mathbb{R}^{k+1} : \sum_i t_i = 1, t_i \geq 0\}.$$

A singular simplex $\sigma : \Delta[k] \rightarrow K^n$ in the Klein model is said to be *straight* if

$$\sigma(t_0, \dots, t_k) = \sum_i t_i \sigma(e_i),$$

where the e_i are the unit vectors in \mathbb{R}^{l+1} . The isometries between the models may then be used to give analogous definitions for straight simplices in B^n and U^n . The image of a straight simplex is the convex hull of the vertices, what we called a *hyperbolic simplex* in the previous chapter. The exact parameterization is not important, except that we have conventionally distinguished a particular parameterization to be straight, uniquely determined by the vertices. A singular simplex $\sigma : \Delta[k] \rightarrow M$ in a hyperbolic manifold M is said to be straight if the lift to the universal cover $\tilde{\sigma} : \Delta[k] \rightarrow \mathbb{H}^n$ is straight.

It is easy to see that the boundary map ∂ takes straight k -simplices to chains of straight $(l-1)$ -simplices, so the chains composed of straight simplices form a chain complex $C_k^{str}(M)$. The inclusion $\iota : C_k^{str}(M) \hookrightarrow C_k(M)$ is a chain map.

In the other direction, we can define a straightening map $\text{Str} : C_k(\mathbb{H}^n) \rightarrow C_k^{str}(\mathbb{H}^n)$ taking a singular simplex $\sigma : \Delta[k] \rightarrow \mathbb{H}^n$ to the unique straight simplex defined by the vertices $\sigma(e_i)$. Again, it is clear that this is a chain map. The more general straightening map $\text{Str} : C_k(M) \rightarrow C_k^{str}(M)$ for a hyperbolic manifold M is defined by

$$\sigma \mapsto \pi \circ \text{Str}(\tilde{\sigma})$$

where $\tilde{\sigma} \in C_k(\mathbb{H}^n)$ is a lift of σ and $\pi : \mathbb{H}^n \rightarrow M$ is the covering map.

Lemma 3.8. *The composition $\iota \circ \text{Str} : C_k(M) \rightarrow C_k(M)$ is chain homotopic to the identity.*

Proof. Every simplex σ is homotopic to $\text{Str}(\sigma)$ by the straight line homotopy. That homotopy is a map $h_\sigma : \Delta[k] \times I \rightarrow M$. By subdividing $\Delta[k] \times I$ into $(k+1)$ -simplices and giving them appropriate signs, h_σ defines a chain in $C_{k+1}(M)$. The map $\sigma \mapsto h_\sigma$ gives the desired chain homotopy. We omit the details as this is essentially the same as the standard proof that homotopic maps of spaces induce the same map on homology, the only difference being that Str is not induced by a global map of spaces. \square

The proofs of Lemma 3.2 and Corollary 3.3 hold without modification for maps on homology induced by chain maps like Str , not only those induced by maps of spaces. The lemma then implies that the Gromov norm of a homology class may be computed as the infimum of the ℓ^1 norm over only the straight chain representatives. With this, the first half of Theorem 3.6 is easy.

Lemma 3.9. *Let M be a closed, connected hyperbolic manifold. Then*

$$\text{vol}(M) \leq v_n \|M\|.$$

Proof. Let dV_M be the volume form on M . Let $a = \sum_i c_i \sigma_i$ be a straight singular chain representing the fundamental class $[M]$.

$$\text{vol}(M) = \int_{[M]} dV_M = \sum_i c_i \int_{\Delta[n]} \sigma_i^*(dV_M) \leq \sum_i |c_i| \text{vol}(\sigma_i) < v_n \|a\|$$

Taking the infimum over all such a gives

$$\text{vol}(M) \leq v_n \|M\|. \quad \square$$

That lemma was easy because it is morally obvious; $\|M\|$ counts (straight) simplices in a triangulation (or is the limit of such counts), each simplex has volume less than the maximal v_n , so the total volume should be less than or equal to the product. To prove the opposite inequality we intuitively need to present singular chains representing the fundamental class using simplices arbitrarily close to regular ideal simplices. We present a proof found in

Ratcliffe ([Rat06], Theorem 11.4.3) which does exactly that, but first we need a few facts about Haar measure on $I(\mathbb{H}^n)$.

Let $G = I(\mathbb{H}^n)$, and $H < G$ be the stabilizer subgroup for a single point x_0 . By homogeneity, we may suppose $x_0 = 0 \in B^n$. By the spherical symmetry of the model B^n , we see that $H \cong O(n)$, and in particular H is compact. Let dh be the left-invariant Haar measure on H , normalized to have total measure 1. The evaluation map $G \mapsto \mathbb{H}^n$, $g \mapsto g(x_0)$ gives a homeomorphism of the coset space $G/H \cong \mathbb{H}^n$. Thus we have a left-invariant measure $d(gH)$ on G/H corresponding to hyperbolic volume. We define left-invariant Haar measure on G by integrating first over H and then over cosets in G/H . That is, given a function $f : G \rightarrow \mathbb{R}$, its integral is defined by

$$\int_G f dg = \int_{G/H} \left(\int_H f(gh) dh \right) d(gH).$$

Lemma 3.10. *The Haar measure dg is bi-invariant.*

Proof. Let $G_+ < G$ be the subgroup of orientation preserving isometries. We will show that G_+ is equal to its own commutator subgroup. Working in the upper half-space model, by Theorem 2.5, G_+ is generated by compositions $\rho_1\rho_2$ of two inversions through spheres (or vertical hyperplanes) S_1 and S_2 orthogonal to ∂U^n . There exists another sphere S_3 orthogonal to ∂U^n which is tangent to both S_1 and S_2 , the points of tangency lying on ∂U^n . Then $\rho_1\rho_2 = (\rho_1\rho_3)(\rho_3\rho_2)$, so it suffices to consider compositions of inversions through tangent spheres. By conjugating, we may move the point of tangency to ∞ , so the spheres S_1 and S_3 are parallel vertical hyperplanes. Then $\tau = \rho_1\rho_3$ is a horizontal translation. Conjugating again, we may assume that τ is translation by e_1 . Let m_2 be dilation by 2. Then

$$m_2\tau m_2^{-1}\tau^{-1}(x) = 2 \left(e_1 + \frac{1}{2}(-e_1 + x) \right) = e_1 + x = \tau(x).$$

Since $\tau = [m_2, \tau]$ is a commutator, G_+ is its own commutator subgroup. The commutator subgroup of G must be contained in G_+ , so it is G_+ . It follows that the abelianization of G is $G_{\text{ab}} = G/G_+ \cong \mathbb{Z}/2$.

Now for any $k \in G$, the right translation $(dg)k$ is again left-invariant. By the uniqueness of Haar measure, this differs from the original measure dg only by a constant scaling factor. We get a group homomorphism called the *modular function* $\Delta : G \rightarrow \mathbb{R}$ defined by $(dg)k = \Delta(k)dg$. Since \mathbb{R} is abelian, Δ factors through $G_{\text{ab}} = \mathbb{Z}/2$, but there are no nontrivial homomorphisms $\mathbb{Z}/2 \rightarrow \mathbb{R}$. Thus Δ is trivial, so dg is bi-invariant. \square

Lemma 3.11. *Given a subset $S \subset \mathbb{H}^n$ and a point $x \in \mathbb{H}^n$, define*

$$\tilde{S} = \{g \in G : g(x) \in S\}.$$

Then \tilde{S} is open (respectively closed) in G if and only if S is open (closed) in \mathbb{H}^n . Additionally $\text{vol}(\tilde{S}) = \text{vol}(S)$.

Proof. The first statement follows simply because \tilde{S} is the preimage of S under the continuous evaluation map $G \rightarrow \mathbb{H}^n$, sending $g \mapsto g(x)$.

For the statement about volumes, first consider $\tilde{S}_0 = \{g \in G : g(x_0) \in S\}$. Let $\chi_{\tilde{S}_0}$ be the indicator function. Note that $\chi_{\tilde{S}_0}$ is invariant under right multiplication by H .

$$\begin{aligned} \text{vol}(\tilde{S}_0) &= \int_G \chi_{\tilde{S}_0}(g) dg \\ &= \int_{G/H} \left(\int_H \chi_{\tilde{S}_0}(gh) dh \right) d(gH) \\ &= \int_{G/H} \chi_{\tilde{S}_0/H}(gH) d(gH) \\ &= \text{vol}(S) \end{aligned}$$

Now choose $g \in G$ such that $g(x) = x_0$. Then $\tilde{S} = \tilde{S}_0 g$, so by the previous lemma

$$\text{vol}(\tilde{S}) = \text{vol}(\tilde{S}_0) = \text{vol}(S). \quad \square$$

Proof of the Proportionality Principle for Hyperbolic Manifolds. Suppose $M = \Gamma \backslash \mathbb{H}^n$ is a closed, connected hyperbolic manifold, with covering map $\pi : \mathbb{H}^n \rightarrow M$. Fix a regular hyperbolic n -simplex $\Delta \subset \mathbb{H}^n$ with side length ℓ and vertices x_0, \dots, x_n , and choose a fundamental domain $P_0 \subset \mathbb{H}^n$ for M such that $x_0 \in \text{int}(P_0)$. Now consider a straight singular simplex $\sigma : \Delta[n] \rightarrow M$ with vertices $\sigma(e_i) = \pi(x_0)$ for all i . There is a unique lift $\tilde{\sigma} : \Delta[n] \rightarrow \mathbb{H}^n$ of σ with $\tilde{\sigma}(e_0) = x_0$. The other vertices of $\tilde{\sigma}$ all lie in the Γ orbit of x_0 , so there are unique isometries $\gamma_i \in \Gamma$ such that $\tilde{\sigma}(e_i) = \gamma_i(x_0)$, where γ_0 is the identity. For each i , we define sets of isometries

$$S_i = \{g \in G : g(x_i) \in \gamma_i(P_0)\}.$$

By Lemma 3.11,

$$\text{vol}(S_i) = \text{vol}(\gamma_i(P_0)) = \text{vol}(M)$$

for each i . Consequently the intersection $S_\sigma = S_0 \cap \dots \cap S_n$ has finite volume.

Now suppose that S_σ is nonempty for a particular σ , containing some isometry g . Then by the triangle inequality,

$$\begin{aligned} d(x_0, \gamma_i(x_0)) &\leq d(x_0, g(x_0)) + d(g(x_0), g(x_i)) + d(g(x_i), \gamma_i(x_0)) \\ &\leq \ell + 2\text{diam}(P_0) \end{aligned}$$

Since Γ acts freely and properly discontinuously, and P_0 has a finite diameter, only finitely many $\gamma \in \Gamma$ satisfy that inequality. Consequently S_σ is nonempty for only finitely many σ . Similarly,

$$\begin{aligned} d(\gamma_i(x_0), \gamma_j(x_0)) &\geq d(g(x_i), g(x_j)) - d(\gamma_i(x_0), g(x_i)) - d(\gamma_j(x_0), g(x_j)) \\ &\geq \ell - 2\text{diam}(P_0) \end{aligned}$$

so for sufficiently large ℓ we may assume that σ is non-degenerate. Let $c_\sigma = \pm \text{vol}(S_\sigma)$, with sign indicating whether σ preserves or reverses orientation. We can define the finite chain $a_\ell = \sum_\sigma c_\sigma \sigma$.

I claim that a_ℓ is a cycle. To see this, we construct a covering chain. Let $\tilde{\sigma}$ be a straight singular simplex in \mathbb{H}^n whose vertices are $\gamma_i(x_0)$ for $\gamma_0, \dots, \gamma_n \in \Gamma$. We then define $S_i, S_{\tilde{\sigma}}$, and $c_{\tilde{\sigma}}$ exactly as above, the only difference being that γ_0 may not be the identity. The chain $\tilde{a}_\ell = \sum_{\tilde{\sigma}} c_{\tilde{\sigma}} \tilde{\sigma}$ is not finite, but it is locally finite. If $\pi(\tilde{\sigma}) = \pi(\tilde{\sigma}')$, then the sets $S_{\tilde{\sigma}}$ and $S_{\tilde{\sigma}'}$ differ only by right translation by an element of Γ , so $c_{\tilde{\sigma}} = c_{\tilde{\sigma}'}$. Thus \tilde{a}_ℓ is exactly the Γ -covering of the chain a_ℓ .

Write the boundary as

$$\partial \tilde{a}_\ell = \sum_i \sum_{\tilde{\sigma}} (-1)^i c_{\tilde{\sigma}} \partial_i \tilde{\sigma} = \sum_{\tilde{\tau}} c_{\tilde{\tau}} \tilde{\tau} \quad (3.1)$$

for straight $(n-1)$ -simplices $\tilde{\tau}$. Let $T = \bigcup_{\gamma \in \Gamma} \partial \gamma(P_0)$ be the boundary between the copies of the fundamental domain tiling \mathbb{H}^n , and define the set of isometries

$$S_T = \{g \in G : g(x_i) \in T \text{ for some } i\}.$$

Given any $g \in G$ with $g \notin S_T$, there is an associated $\tilde{\sigma}_g$ whose vertices are the points of Γx_0 in the same fundamental domains as the points $g(x_i)$. For $0 \leq i \leq n$, define the sets of isometries

$$S_{\tilde{\tau}_+}^i = \left\{ g \in G : \begin{array}{l} g \notin S_T \\ \partial_i \tilde{\sigma}_g = \tilde{\tau} \\ \tilde{\sigma}_g \text{ orientation preserving} \end{array} \right\}$$

That is, $S_{\tilde{\tau}_+}^i$ is the subset of isometries which account for the positive coefficients $c_{\tilde{\sigma}}$ in the i -th summand in equation (3.1) which contribute to $c_{\tilde{\tau}}$. Define $S_{\tilde{\tau}_-}^i$ identically, but with $\tilde{\sigma}_g$ orientation reversing. Let ρ be the reflection through the hyperplane containing $x_0, \dots, \hat{x}_i, \dots, x_n$. For sufficiently large ℓ (in comparison to the diameter of P_0), it is clear that $\tilde{\sigma}_g$ and $\tilde{\sigma}_{g\rho}$ have opposite orientations. Since ρ preserves the i -th face of Δ , it follows that the symmetric difference of $S_{\tilde{\tau}_-}^i$ and the right translate $S_{\tilde{\tau}_+}^i \rho$ is contained in S_T . Since $\text{vol}(T) = 0$, by Lemma 3.11 we have $\text{vol}(S_{\tilde{\tau}_+}^i) = \text{vol}(S_{\tilde{\tau}_-}^i)$. The two volumes cancel, so the i -th summand of equation (3.1) contributes 0 to $c_{\tilde{\tau}}$. Since this is true for all i , we conclude that $c_{\tilde{\tau}} = 0$, so \tilde{a}_ℓ is a cycle, and consequently a_ℓ is a cycle.

The cycle has norm

$$\begin{aligned} \|a_\ell\|_1 &= \sum_{\sigma} |c_\sigma| \\ &= \text{vol} \left(\bigcup_{\sigma} S_\sigma \right) \\ &= \text{vol}(S_0) \\ &= \text{vol}(M) \end{aligned}$$

The fundamental class $[M]$ generates $H_n(M)$, so $[a_\ell] = k_\ell [M]$ for some constant k_ℓ , which can be determined by integrating against the volume form.

$$\begin{aligned} \int_{[M]} dV &= \text{vol}(M) \\ \int_{a_\ell} dV &= \sum_{\sigma} c_\sigma \int_{\Delta[n]} \tilde{\sigma}^*(dV) \\ &= \sum_{\sigma} |c_\sigma| \text{vol}(\tilde{\sigma}) \\ &\geq \text{vol}(M) \cdot \min_{c_\sigma \neq 0} \text{vol}(\tilde{\sigma}) \\ k_\ell &\geq \min_{c_\sigma \neq 0} \text{vol}(\tilde{\sigma}) \end{aligned}$$

When $c_\sigma \neq 0$, there is an isometry g such that $d(g(x_i), \tilde{\sigma}(e_i)) < \text{diam}(P_0)$. For large ℓ , this means that $\tilde{\sigma}$ is very well approximated by a regular simplex of side length ℓ . As ℓ increases we have $\lim_{\ell \rightarrow \infty} \text{vol}(\tilde{\sigma}) = v_n$, and consequently

$$\lim_{\ell \rightarrow \infty} k_\ell \geq v_n.$$

Finally, we get an upper bound on the simplicial volume.

$$\|M\| = \frac{\|[a_\ell]\|}{k_\ell} \leq \lim_{\ell \rightarrow \infty} \frac{\|a_\ell\|_1}{k_\ell} \leq \frac{\text{vol}(M)}{v_n}$$

Together with Lemma 3.9, this proves

$$\frac{\text{vol}(M)}{\|M\|} = v_n. \quad \square$$

3.3. Measure Homology and the General Proportionality Principle.

The proof of the proportionality principle in the previous section has the advantage of being direct and visual, but was a bit messier than one might hope for. In fact there is a much cleaner proof, based on the same basic idea, if we switch from singular homology to measure homology, an equivalent theory due to Thurston. Moreover, measure homology will help us prove a more general proportionality principle, and will be used in the proof of Mostow rigidity in the next chapter.

Throughout this section $C_k(M)$ should be understood to contain only chains of C^1 singular simplices.

Definition 3.12. Given a manifold M , let μ be a Borel measure on the mapping space $C^1(\Delta[k], M)$, given the C^1 topology. (We use the C^1 topology rather than the coarser compact-open topology so that the various maps of $C^1(\Delta[k], M)$ are continuous. We will omit the details of checking this, as it is technical and entirely unilluminating.) Every measure has a unique Jordan decomposition $\mu = \mu_+ - \mu_-$, where μ_+ and μ_- are positive measures. The *total variation* of μ is then defined as

$$\|\mu\|_{tv} = \int_{C^1(\Delta[k], M)} d\mu_+ + \int_{C^1(\Delta[k], M)} d\mu_-.$$

Equivalently,

$$\|\mu\|_{tv} = \sup_{|f| < 1} \int f d\mu.$$

Define $\mathcal{C}_k(M)$ to be the set of Borel measures on $C^1(\Delta[k], M)$ with compact support and finite total variation. The face restrictions $\partial_i : C_k(M) \rightarrow C_{k-1}(M)$ give push-forward maps which we abusively give the same names, $\partial_i : \mathcal{C}_k(M) \rightarrow \mathcal{C}_{k-1}(M)$.

$$(\partial_i)_* \mu(B) = \mu((\partial_i)^{-1}(B))$$

The reader may easily check that the differential $\partial = \sum_i (-1)^i \partial_i : \mathcal{C}_k(M) \rightarrow \mathcal{C}_{k-1}(M)$ satisfies $\partial^2 = 0$, so it makes $\mathcal{C}(M)$ into a chain complex. The homology of this chain complex is the *measure homology* of M , denoted $\mathcal{H}_k(M)$. In analogy to the Gromov norm, we have a seminorm $\|\cdot\|_{mh} : \mathcal{H}_k(M) \rightarrow \mathbb{R}$ defined by

$$\|\alpha\|_{mh} = \inf_{[\mu] = \alpha} \|\mu\|_{tv}.$$

Just as we could integrate closed differential forms over singular homology classes, well defined by Stokes' theorem, we can analogously integrate forms over measure homology classes. For $\mu \in \mathcal{C}_k(M)$ and $\omega \in \Omega^k(M)$,

$$\int_{\mu} \omega = \int_{C^1(\Delta[k], M)} \left(\int_{\Delta[k]} \sigma^* \omega \right) d\mu.$$

We check that this is well defined on homology, for ω closed, which still comes down to Stokes' theorem.

$$\begin{aligned} \int_{\partial\mu} \omega &= \sum_i (-1)^i \int_{\tau \in C^1(\Delta[k-1], M)} \left(\int_{\tau} \omega \right) d(\partial_i \mu) \\ &= \int_{\sigma \in C^1(\Delta[k], M)} \left(\sum_i (-1)^i \int_{\partial_i \sigma} \omega \right) d\mu \\ &= \int_{\sigma \in C^1(\Delta[k], M)} \left(\int_{\partial\sigma} \omega \right) d\mu \\ &= \int_{\sigma \in C^1(\Delta[k], M)} \left(\int_{\sigma} d\omega \right) d\mu \\ &= 0 \end{aligned}$$

Given $\sigma \in C^1(\Delta[k], M)$, there is a corresponding atomic measure $\mu_{\sigma} \in \mathcal{C}_k(M)$.

$$\mu_{\sigma}(B) = \begin{cases} 1 & \sigma \in B \\ 0 & \sigma \notin B \end{cases}$$

This gives an inclusion map $\iota : \mathcal{C}_k(M) \rightarrow \mathcal{C}_k(M)$.

$$\iota \left(\sum_i c_i \sigma_i \right) = \sum_i c_i \mu_{\sigma_i}$$

ι is easily seen to be a chain map. We also note that the integral of a differential form ω over the measure $\iota(\sigma)$ as defined above is the same as the simple integral of ω over σ . The following theorem due to Zastrow [Zas98] and Löh [L05] justifies the parallel we are making between measure homology and singular homology.

Theorem 3.13. *The inclusion of atomic measures $\iota : \mathcal{C}_*(M) \rightarrow \mathcal{C}_*(M)$ induces an isometric isomorphism $(H_*(M), \|\cdot\|) \cong (\mathcal{H}_*(M), \|\cdot\|_{mh})$.*

In view of the theorem, we define the fundamental class in measure homology as $[M]_{mh} = \iota_*([M])$, and the simplicial volume as $\|M\|_{mh} = \|[M]_{mh}\|_{mh}$. We proceed to reprove the proportionality principle in this context.

Measures may be straightened by pushing forward along the straightening map for simplices.

$$\text{Str}(\mu)(B) = \mu(\text{Str}^{-1}(B))$$

That is, straight measures are those with support contained in the subspace of straight singular simplices. The reasoning from the case of singular chains may be ported over to show that $\text{Str} : \mathcal{C}_*(M) \rightarrow \mathcal{C}_*(M)$ is a chain map and is chain homotopic to the identity; we leave the details to the reader. Thus we may restrict our attention to straight measures

when computing the total variation seminorm. With this, the proof of Lemma 3.9 carries over immediately.

Lemma 3.14. *Let M be a closed, connected hyperbolic manifold. Then*

$$\text{vol}(M) \leq v_n \|M\|_{mh}$$

Proof. Let dV_M be the volume form on M , and let μ be a straight measure representing $[M]_{mh}$.

$$\text{vol}(M) = \int_{\mu} dV_M = \int_{C^1(\Delta[k], M)} \left(\int_{\sigma} dV_M \right) d\mu \leq v_n \|\mu\|_{tv}$$

Taking the infimum over straight representatives μ gives the result. \square

For the more difficult inequality, we need to introduce smearing measures. Recall the Haar measure dg on $G = I(\mathbb{H}^n)$ defined in the previous section. We restrict our attention to the orientation preserving isometry group G_+ and define the Haar measure dg in the same way, except renormalizing so that H_+ (the stabilizer of a point within G_+) has measure 1 instead of H . (This renormalization follows the convention used in Ratcliffe and Thurston, and makes no difference other than a factor of 2.) As before the Haar measure is bi-invariant, so it descends to a measure on $\Gamma \backslash G_+$ for any discrete isometry group Γ .

Fix a C^1 singular simplex $\sigma : \Delta[k] \rightarrow \mathbb{H}^n$. Intuitively, the *smear* of σ is the uniform measure on the isometric copies of σ projected down to M . Formally, we have a translation map $\Gamma \backslash G_+ \rightarrow C^1(\Delta[k], M)$ taking $g \mapsto \pi \circ g \circ \sigma$, and $\text{smear}(\sigma)$ is the push-forward of dg by this map. Since $\text{smear}(\sigma)$ is equally spread over all the isometric copies of σ , $\text{smear}(f\sigma) = \text{smear}(\sigma)$ for any isometry $f \in G_+$. Thus we may talk about the smear of a simplex $\sigma \in C^1(\Delta[k], M)$ by first lifting to $C^1(\Delta[k], \mathbb{H}^n)$ and then smearing back down. Moreover, we may extend by linearity to a map $\text{smear} : C_k(M) \rightarrow C_k(M)$, easily seen to be a chain map.

Lemma 3.15. *For any singular simplex σ ,*

$$\|\text{smear}(\sigma)\|_{tv} = \text{vol}(M).$$

Proof. The smear is a nonnegative measure.

$$\|\text{smear}(\sigma)\|_{tv} = \int_{C^1(\Delta[k], M)} d(\text{smear}(\sigma)) = \int_{\Gamma \backslash G_+} d(\Gamma g) = \text{vol}(M) \quad \square$$

Unfortunately $\text{smear}(\sigma)$ is not a cycle. To fix this, we require that σ be a straight simplex, let ρ be a reflection through some hyperplane in \mathbb{H}^n , and define

$$\text{avg}(\sigma) = \frac{1}{2}(\text{smear}(\sigma) - \text{smear}(\rho\sigma)).$$

Then $\text{avg}(\sigma)$ is a cycle, because for any $\tau \in C_{k-1}(M)$, lifted to $\tilde{\tau} \in C_{k-1}(\mathbb{H}^n)$, the sets of orientation preserving and orientation reversing isometries g such that $\tilde{\tau}$ is a face of $g\sigma$ are precisely related by reflection through the hyperplane containing $\tilde{\tau}$, so they cancel out. $\text{Smear}(\sigma)$ is supported only on simplices with the same orientation as σ , and $\text{smear}(\rho\sigma)$ is supported only on those with opposite orientation, so their total variations add.

$$\|\text{avg}(\sigma)\|_{tv} = \text{vol}(M)$$

Theorem 3.16 (Proportionality Principle for Hyperbolic Manifolds (Measure Homology)). *Let M be a closed, connected hyperbolic manifold. Then*

$$\text{vol}(M) = v_n \|M\|_{mh}.$$

Proof. Fix a straight singular simplex $\sigma_0 \in C^1(\Delta[n], M)$. We integrate against the volume form to determine the homology class of $\text{avg}(\sigma_0)$.

$$\int_{C^1(\Delta[n], M)} \left(\int_{\Delta[n]} \sigma^* dV_M \right) d(\text{avg}(\sigma_0)) = \text{vol}(\sigma_0) \|\text{avg}(\sigma_0)\| = \text{vol}(\sigma_0) \text{vol}(M) \quad (3.2)$$

Thus $[\text{avg}(\sigma_0)] = \text{vol}(\sigma_0)[M]_{mh}$. Now we take an infimum.

$$\|M\|_{mh} \leq \inf_{\sigma_0} \frac{\|\text{avg}(\sigma_0)\|_{tv}}{\text{vol}(\sigma_0)} = \frac{\text{vol}(M)}{v_n}$$

Together with Lemma 3.14, this completes the proof. \square

Smear measures actually let us prove a more general result, assuming Theorem 3.13.

Theorem 3.17 (General Proportionality Principle for Simplicial Volume). *Let M and N be closed, connected Riemannian manifolds with isometric universal covers. Then*

$$\frac{\|M\|}{\text{vol}(M)} = \frac{\|N\|}{\text{vol}(N)}.$$

We defined smear measures specifically for hyperbolic manifolds, but the same definition works much more generally, for any manifold $M = \Gamma \backslash X$ where X is a homogenous, orientable, Riemannian manifold with compact isotropy subgroups, and Γ is a discrete subgroup of isometries acting freely and properly discontinuously. Lemma 3.15 holds in this more general context, and the triangle inequality for the total variation seminorm implies the following extension.

Lemma 3.18. *For any singular chain $a \in C_k(M)$,*

$$\|\text{smear}(a)\|_{tv} \leq \text{vol}(M) \|a\|_1. \quad \square$$

Proof of the General Proportionality Principle. Consider the chain map $\text{smear}_{M,N} : C_k(M) \rightarrow C_k(N)$ which takes a singular chain on M , lifts to the universal cover, and smears back down to a measure on $C^1(\Delta[k], N)$. Let $a \in C_n(M)$ represent the fundamental class $[M]$. By the previous lemma, suitably adapted,

$$\|\text{smear}_{M,N}(a)\|_{tv} \leq \text{vol}(N) \|a\|_1.$$

On the other hand, since $\text{smear}_{M,N}$ is a chain map, $\text{smear}_{M,N}(a)$ is a cycle, some multiple of $[N]_{mh}$. We copy the integration from equation (3.2) for $a = \sum_i c_i \sigma_i$.

$$\begin{aligned} \int_{\text{smear}_{M,N}(a)} dV_N &= \sum_i c_i \int_{C^1(\Delta[n], N)} \left(\int_{\Delta[n]} \sigma^* dV_N \right) d(\text{smear}_{M,N}(\sigma_i)) \\ &= \left(\sum_i c_i \int_{\Delta[n]} \sigma_i^* dV_N \right) \text{vol}(N) \\ &= \text{vol}(M) \text{vol}(N) \\ [\text{smear}_{M,N}(a)] &= \text{vol}(M) [N]_{mh} \end{aligned}$$

Together with the previous equation,

$$\text{vol}(M) \|N\|_{mh} = \|\text{smear}_{M,N}(a)\|_{tv} \leq \text{vol}(N) \|a\|_1.$$

Finally we take the infimum over all representatives a for $[M]$ and apply Theorem 3.13.

$$\frac{\|N\|}{\text{vol}(N)} \leq \frac{\|M\|}{\text{vol}(M)}$$

Reversing the roles of M and N gives the reverse inequality, so in fact

$$\frac{\|N\|}{\text{vol}(N)} = \frac{\|M\|}{\text{vol}(M)} \quad \square$$

4. MOSTOW RIGIDITY

The proportionality principle for hyperbolic manifolds, proven in the previous chapter, is remarkable because it reduces geometry to topology. It states that the volume of a closed hyperbolic manifold, an inherently geometric quantity, can be computed solely from the singular homology, which is purely topological. It implies that the volume of a closed hyperbolic manifold is a homotopy invariant, independent of the particular hyperbolic metric. In this chapter we will prove the Mostow rigidity theorem, which tells us that not only the volume but *all* of the geometric structure of a closed hyperbolic manifold is uniquely determined by its fundamental group.

Theorem 4.1 (Mostow Rigidity). *Let $M = \Gamma \backslash \mathbb{H}^n$ and $N = \Xi \backslash \mathbb{H}^n$ be closed hyperbolic manifolds, for $n \geq 3$. If $\pi_1(M) \cong \pi_1(N)$, then M and N are isometric.*

The proof we present is originally due to Mostow and Gromov, but we follow the treatments in sections 5.9 and 6.3 of Thurston [Thu02] and in section 11.6 of Ratcliffe [Rat06].

4.1. Obtain and Lift a Homotopy Equivalence.

First, recall that covering maps induce isomorphisms on all homotopy groups π_i for $i \geq 2$ ([Hat02], Proposition 4.1). Since \mathbb{H}^n is homotopically trivial, this implies that a hyperbolic manifold $M = \Gamma \backslash \mathbb{H}^n$ is an Eilenberg-MacLane space $K(\Gamma, 1)$. Since Eilenberg-MacLane spaces are unique up to homotopy equivalence ([Hat02], Theorem 1B.8 for the particular case of $K(G, 1)$, Proposition 4.30 in general), the isomorphism of fundamental groups in fact guarantees a homotopy equivalence $\phi : M \rightarrow N$. We then lift ϕ to a map $\tilde{\phi} : \mathbb{H}^n \rightarrow \mathbb{H}^n$ such that the following square commutes,

$$\begin{array}{ccc} \mathbb{H}^n & \xrightarrow{\tilde{\phi}} & \mathbb{H}^n \\ \pi \downarrow & & \downarrow \eta \\ M & \xrightarrow{\phi} & N \end{array}$$

where π and η are the covering projections. Let $\psi : N \rightarrow M$ be a homotopy inverse to ϕ , and $F : M \times [0, 1] \rightarrow M$ a homotopy between $\psi\phi$ and id_M . We lift ψ to $\tilde{\psi} : \mathbb{H}^n \rightarrow \mathbb{H}^n$, and then F lifts to a homotopy \tilde{F} between $\tilde{\psi}\tilde{\phi}$ and some $\gamma \in \Gamma$ (a lift of id_M). By replacing $\tilde{\psi}$ and \tilde{F} by $\gamma^{-1}\tilde{\psi}$ and $\gamma^{-1}\tilde{F}$, we may assume that $\tilde{F}_1 = id_{\mathbb{H}^n}$. For any $\gamma \in \Gamma$, consider the following.

$$\begin{aligned} \pi\tilde{F}(\gamma \times id) &= F(\pi \times id)(\gamma \times id) \\ &= F(\pi \times id) \\ &= \pi\tilde{F} \end{aligned}$$

Thus $\tilde{F}(\gamma \times id) = \gamma' \tilde{F}$ for some $\gamma' \in \Gamma$. Since $\tilde{F}_1 = id_{\mathbb{H}^n}$, we must have $\gamma' = \gamma$, so \tilde{F} is Γ -equivariant. In particular, $\tilde{\psi}\tilde{\phi}$ is Γ -equivariant.

4.2. Extend to $\partial\mathbb{H}^n$.

Recall that isometries of \mathbb{H}^n extend to continuous maps of $\partial\mathbb{H}^n$ because they map geodesics to geodesics, preserving the property of being asymptotic. We want to show that $\tilde{\phi}$ is close enough to being an isometry that it may be similarly extended to $\partial\mathbb{H}^n$. This part of the proof is principally analytic. In the interest of time, we will occasionally skip over some of the messier details; see Ratcliffe for a more complete treatment.

It is a basic fact of differential topology that every continuous map is homotopic to a smooth map, so we may assume that ϕ and ψ are smooth. In particular, as C^1 maps between compact Riemannian manifolds, they are Lipschitz. Let k be a Lipschitz constant which works for both. The lifts $\tilde{\phi}$ and $\tilde{\psi}$ behave identically to ϕ and ψ locally. That is, if we choose ϵ such that each ϵ -neighborhood in M and N is evenly covered by ϵ -neighborhoods in \mathbb{H}^n , then for any points $x, y \in \mathbb{H}^n$ with $d(x, y) < \epsilon$ we have

$$d(\tilde{\phi}(x), \tilde{\phi}(y)) \leq k d(x, y),$$

and also for $\tilde{\psi}$. For general $x, y \in \mathbb{H}^n$, the geodesic segment connecting x and y may be split into segments of length less than ϵ , each of which is stretched by less than a factor of k . Thus $\tilde{\phi}$ and $\tilde{\psi}$ are also Lipschitz, with the same constant k . We will show that they satisfy a stronger condition.

Definition 4.2. Given a metric space X , a map $f : X \rightarrow X$ is a *pseudo-isometry* if there exist constants k and ℓ such that for all $x, y \in X$,

$$k^{-1}d(x, y) - \ell \leq d(f(x), f(y)) \leq k d(x, y).$$

Let $P \subset \mathbb{H}^n$ be a fundamental domain for M . Since \bar{P} is compact, we have a finite diameter $b = \text{diam}(\bar{P} \times [0, 1])$. In particular, for any $x \in \bar{P}$,

$$d(x, \tilde{\psi}\tilde{\phi}(x)) \leq b.$$

Since \tilde{F} is Γ -equivariant, this in fact holds for all $x \in \mathbb{H}^n$. For any $x, y \in \mathbb{H}^n$, we have by the triangle inequality

$$\begin{aligned} d(x, y) &\leq d(x, \tilde{\psi}\tilde{\phi}(x)) + d(\tilde{\psi}\tilde{\phi}(x), \tilde{\psi}\tilde{\phi}(y)) + d(\tilde{\psi}\tilde{\phi}(y), y) \\ &\leq 2b + k d(\tilde{\phi}(x), \tilde{\phi}(y)) \end{aligned}$$

$$k^{-1}d(x, y) - \frac{2b}{k} \leq d(\tilde{\phi}(x), \tilde{\phi}(y)) \leq k d(x, y)$$

Thus $\tilde{\phi}$ is a pseudo-isometry. We now prove a series of lemmas investigating the behavior of pseudo-isometries of hyperbolic space.

Lemma 4.3. *Let $L \subset \mathbb{H}^n$ be a geodesic line, $\rho : \mathbb{H}^n \rightarrow L$ be the orthogonal projection onto the line, and $\beta : [t_0, t_1] \rightarrow \mathbb{H}^n$ be a finite length path. If δ is the distance from L to the image of β , then*

$$\text{length}(\rho\beta) \leq \frac{1}{\cosh \delta} \text{length}(\beta).$$

This lemma reflects the fact that the boundary surface of a δ -neighborhood of a geodesic in \mathbb{H}^n , a hyperbolic analogue of a cylinder with constant radius, is positively curved. This is tightly related to the fact that any two disjoint geodesics in \mathbb{H}^n diverge, the distance

between them increasing without bound as they go along. Contrast this to the Euclidean case, where cylinders have 0 curvature and parallel lines remain the same distance apart at all points.

Proof. Without loss of generality, we let $L = \mathbb{R}_{>0}e_n \subset U^n$. Hyperplanes perpendicular to L are then hemispheres centered around the origin, so the projection is $\rho(x) = |x|e_n$. One may verify by direct computation that

$$\cosh(d(x, \rho(x))) = |x|/x_n.$$

The length of the projected path then satisfies

$$\begin{aligned} \text{length}(\rho\beta) &= \int_{t_0}^{t_1} \frac{|(\rho\beta)'(t)|}{\rho\beta(t)_n} dt \\ &= \int_{t_0}^{t_1} \frac{|\beta'(t) \cdot \beta(t)|/|\beta(t)|}{|\beta(t)|} dt \\ &\leq \int_{t_0}^{t_1} \frac{|\beta'(t)|}{|\beta(t)|} dt \\ &\leq \int_{t_0}^{t_1} \frac{|\beta'(t)|}{\cosh(\delta)\beta(t)_n} dt \\ &= \frac{1}{\cosh \delta} \text{length}(\beta) \quad \square \end{aligned}$$

Lemma 4.4. *Let $f : \mathbb{H}^n \rightarrow \mathbb{H}^n$ be a pseudo-isometry with constants k and ℓ , and let $g \subset \mathbb{H}^n$ be a geodesic. Then for sufficiently large δ (defined precisely in the proof in terms of k and ℓ), there exists a constant c such that for any geodesic L with δ -neighborhood denoted $N_\delta(L)$, every bounded component of $g - f^{-1}(N_\delta(L))$ has length less than c .*

The rough idea here is that a pseudo-isometry should roughly preserve arc length, neither stretching nor shrinking it too much. As suggested by the previous lemma, the boundary $\partial N_\delta(L)$ is positively curved, with higher curvature for higher δ . Given two points on $\partial N_\delta(L)$, sufficiently far apart from each other and for sufficiently large δ , a path connecting those points which stays outside of $N_\delta(L)$ is necessarily very inefficient, and thus cannot be the image of a geodesic under a pseudo-isometry.

Proof. We parameterize g , abusing notation, with a map $g : \mathbb{R} \rightarrow \mathbb{H}^n$. Suppose that $fg(t_0)$ and $fg(t_1)$ lie on $\partial N_\delta(L)$. Let ρ be the orthogonal projection onto L . By connectedness, $\rho fg([t_0, t_1])$ must contain the segment of L connecting $\rho fg(t_0)$ and $\rho fg(t_1)$. By the triangle inequality, and also using the previous lemma,

$$\begin{aligned} d(fg(t_0), fg(t_1)) &\leq d(fg(t_0), \rho fg(t_0)) + d(\rho fg(t_0), \rho fg(t_1)) + d(\rho fg(t_1), fg(t_1)) \\ &\leq 2\delta + \cosh(\delta)^{-1} \text{length}(fg([t_0, t_1])) \end{aligned}$$

Applying the fact that f is a pseudo-isometry, we get

$$\begin{aligned} k^{-1}d(g(t_0), g(t_1)) - \ell &\leq d(fg(t_0), fg(t_1)) \leq 2\delta + \frac{k}{\cosh \delta} d(g(t_0), g(t_1)) \\ \left(\frac{1}{k} - \frac{k}{\cosh \delta}\right) d(g(t_0), g(t_1)) &\leq 2\delta + \ell \end{aligned}$$

Thus, for δ sufficiently large such that $1/k > k/\cosh \delta$, the length of the original segment on g is bounded by

$$d(g(t_0), g(t_1)) \leq c = \frac{2\delta + \ell}{\frac{1}{k} - \frac{k}{\cosh \delta}} \quad \square$$

Setting $r = \delta + ck/2$, we immediately have a slightly cleaner corollary.

Corollary 4.5. *If $fg(t_0)$ and $fg(t_1)$ lie inside the closed neighborhood $\overline{N_\delta(L)}$, then the entire path $fg([t_0, t_1])$ lies within $N_r(L)$.* \square

Lemma 4.6. *Given a pseudo-isometry $f : \mathbb{H}^n \rightarrow \mathbb{H}^n$ and a geodesic $g \subset \mathbb{H}^n$, there exists a unique geodesic L_g such that $f(g) \subset N_r(L_g)$.*

Proof. Again, we abuse notation and parameterize the geodesic by a map $g : \mathbb{R} \rightarrow \mathbb{H}^n$. Let L_i be the geodesic line through $fg(-i)$ and $fg(i)$. (Note that $fg(-i) \neq fg(i)$ for sufficiently large i because f is a pseudo-isometry.) By the corollary above, $fg([-i, i]) \subset N_r(L_i)$ for all i . This implies that the lines L_i converge to a geodesic L_g ; we will describe the relevant picture and leave the reader to check any lingering analytic details.

In the Poincaré ball model, the neighborhoods $N_r(L_i)$ are banana-shaped tubes surrounding the geodesics L_i , tapering to pointed ends at the ideal endpoints. Suppose $j > i$. Then $fg([-i, i])$ is contained within both $N_r(L_i)$ and $N_r(L_j)$, so L_i and L_j are within distance $2r$ within this interval. Let H_i^- and H_i^+ be the hyperplanes perpendicular to L_i through the points $fg(-i)$ and $fg(i)$ respectively. Then let $N_{2r}^{[-i, i]}(L_i)$ be the subset of $N_{2r}(L_i)$ lying between H_i^- and H_i^+ . The geodesic L_j must pass through the truncated tapered tube $N_{2r}^{[-i, i]}(L_i)$ from end to end, without intersecting its cylindrical boundary component. For large i , the truncation of the tube neighborhood by the planes H_i^\pm occurs very close to the boundary sphere ∂B^n , where $N_{2r}(L_i)$ is very narrow. One may see that in order for L_j to pass all the way through $N_{2r}^{[-i, i]}(L_i)$, the endpoints of L_j must lie in small neighborhoods U_i^\pm of the endpoints of L_i on ∂B^n . As i increases, the truncation occurs closer to the boundary sphere where the tube narrows even further, and the neighborhoods U_i^\pm shrink correspondingly. Thus the endpoints of the lines L_i form a Cauchy sequence on ∂B^n , and the limits of those Cauchy sequences are the endpoints of the limiting line L_g .

Since L_g is the limit of the L_i , it follows that $f(g) \subset N_r(L_g)$. Furthermore, any two distinct geodesics diverge in \mathbb{H}^n , becoming arbitrarily far apart as they limit towards distinct ideal endpoints, so the line L_g is unique. \square

Corollary 4.7. *A pseudo-isometry $f : \mathbb{H}^n \rightarrow \mathbb{H}^n$ extends to an injective map $f_\infty : \partial\mathbb{H}^n \rightarrow \partial\mathbb{H}^n$.*

Proof. The key point is that the association of geodesics $g \mapsto L_g$ obtained in the previous lemma preserves asymptotic geodesics. If $g_1, g_2 : \mathbb{R} \rightarrow \mathbb{H}^n$ are asymptotic geodesics, then, reparameterizing if necessary, we have $d(g_1(i), g_2(i)) \rightarrow 0$ as $i \rightarrow \infty$. Since f is a pseudo-isometry, we still have $d(fg_1(i), fg_2(i)) \rightarrow 0$ after mapping. Any two distinct ideal points are infinitely far apart, so the fact that $f(g_i) \subset N_r(L_{g_i})$ for both $i = 1, 2$ requires L_{g_1} and L_{g_2} to be asymptotic.

A point $x \in \partial\mathbb{H}^n$ may be identified as the positive endpoint of some geodesic g , that is $x = \lim_{t \rightarrow \infty} g(t)$. We then define $f_\infty(x)$ to be the positive endpoint of L_g . By the above, this is well defined independent of the choice of g . Any two distinct points $x, y \in \partial\mathbb{H}^n$ are the endpoints of some geodesic g , and subsequently $f_\infty(x)$ and $f_\infty(y)$ are the distinct endpoints of L_g . Thus f_∞ is injective. \square

Lemma 4.8. *Let $g \subset \mathbb{H}^n$ be a geodesic, $H \subset \mathbb{H}^n$ be a hyperplane perpendicular to g , and $f : \mathbb{H}^n \rightarrow \mathbb{H}^n$ a pseudo-isometry. Then there exists a constant c , in terms of the pseudo-isometry constants k and ℓ , such that the orthogonal projection of $f(H)$ onto L_g has diameter less than c .*

The constant c here should not be confused with the constant from Lemma 4.4.

Proof. Let $x = g \cap H$, $R \subset H$ be any geodesic through x , and z be one of the ideal endpoints of R . Let K_1 and K_2 be the geodesic lines connecting z to the endpoints of g . Let A_i be the shortest geodesic segments connecting x and the K_i , with endpoints $p_i = A_i \cap K_i$. Since R is perpendicular to g , there is a symmetry in this setup, so $\text{length}(A_1) = \text{length}(A_2) = a$. (With a bit of hyperbolic trigonometry one may see that $a = \cosh^{-1}(\sqrt{2})$.)

Now consider the image of this entire picture under f . Let ρ be the orthogonal projection map onto L_g . Draw the geodesic L_\perp through $f_\infty(z)$ and $\rho(f_\infty(z))$. By the triangle inequality,

$$\begin{aligned} d(\rho(f(x)), L_{K_i}) &\leq d(\rho(f(x)), f(x)) + \text{length}(f(A_i)) + d(f(p_i), L_{K_i}) \\ &\leq 2r + ak = c_1 \end{aligned}$$

where we have used Lemma 4.6 and the fact that f is a pseudo-isomorphism. The shortest segment from $\rho(f(x))$ to one of the lines L_{K_i} must cross L_\perp , so it follows that

$$d(\rho(f(x)), \rho(f_\infty(z))) \leq c_1.$$

Now let $y \in H$ be any other point, $Q \subset H$ be a geodesic through y , and q_1, q_2 be the ideal endpoints of Q . By the above, $d(\rho(f(x)), \rho(f_\infty(q_i))) \leq c_1$. The projection $\rho(L_Q)$ is simply the segment of L_g connecting the points $\rho(f_\infty(q_i))$. Since $d(f(y), L_Q) < r$, and the projection ρ does not increase distances (Lemma 4.3), we conclude that

$$d(\rho(f(y)), \rho(f(x))) \leq c_1 + r = c_2,$$

and subsequently

$$\text{diam}(\rho(f(H))) \leq 2c_2 = c. \quad \square$$

Corollary 4.9. $f_\infty : \partial\mathbb{H}^n \rightarrow \partial\mathbb{H}^n$ is continuous.

Proof. Let $x \in \partial\mathbb{H}^n$, and g be a geodesic limiting to x . Then $f_\infty(x)$ is by definition an endpoint of L_g . Let ρ be the orthogonal projection to L_g . Each hyperplane H perpendicular to L_g defines a half-space neighborhood of $f_\infty(x)$ which we denote H_+ . Any neighborhood of $f_\infty(x)$ contains some such half-space. For a given H_+ , there exists $y \in g$ such that $\rho(f(y)) \in H_+$ and $d(\rho(f(y)), H) > c$, where c is the constant from the previous lemma. If K is the hyperplane through y perpendicular to g and K_+ is the corresponding half-space neighborhood of x , then f maps K_+ into H_+ . Thus f_∞ is continuous at x . \square

4.3. Rigidity.

The steps in the previous section are common to all proofs of Mostow rigidity. There are various ways to proceed from here. We present a particularly quick and elegant proof due to Gromov, using the simplicial volume.

We return to our earlier notation: $M = \Gamma \backslash \mathbb{H}^n$ and $N = \Xi \backslash \mathbb{H}^n$ are hyperbolic manifolds with $n \geq 3$, π and η are the projections from \mathbb{H}^n to M and N respectively, $\phi : M \rightarrow N$ is a homotopy equivalence, and $\tilde{\phi}$ is a lift of ϕ to \mathbb{H}^n . As seen in the previous section, $\tilde{\phi}$ is a pseudo-isometry and thus extends to a continuous map $\tilde{\phi}_\infty$ of the sphere at infinity.

Lemma 4.10. *Let $p_0, \dots, p_n \in \partial\mathbb{H}^n$ be the vertices of a regular ideal simplex. Then the points $\tilde{\phi}_\infty(p_0), \dots, \tilde{\phi}_\infty(p_n)$ are again the vertices of a regular ideal simplex.*

Proof. Suppose not. Then there exist neighborhoods $U_i \subset \mathbb{H}^n$ of the vertices p_i and some $\epsilon > 0$ such that if a straight singular simplex σ has vertices $\sigma(e_i) \in U_i$, then

$$\text{vol}(\text{Str}(\tilde{\phi}\sigma)) < v_n - \epsilon.$$

Fix one such σ_0 . Let G_+ be the group of orientation preserving isometries of \mathbb{H}^n as in section 3.3, and define the subset

$$\tilde{U} = \{g \in G_+ : g(\sigma_0(e_i)) \in U_i \forall i\}.$$

By Lemma 3.11, \tilde{U} is open in G_+ and thus has positive measure.

As computed in equation (3.2), $[\text{avg}(\sigma_0)] = \text{vol}(\sigma_0)[M]_{mh}$. Since ϕ is a homotopy equivalence and thus has degree 1, and Str induces the identity map on homology,

$$[\text{Str}(\phi_* \text{avg}(\sigma_0))] = \text{vol}(\sigma_0)[N]_{mh}.$$

On the other hand, we can directly integrate the volume form dV_N over the measure $\text{Str}(\phi_* \text{avg}(\sigma_0))$ to determine its homology class.

$$\begin{aligned} \int_{\text{Str}(\phi_* \text{smear}(\sigma_0))} dV_N &= \int_{C^1(\Delta[n], N)} \left(\int_{\text{Str}(\phi\sigma)} dV_N \right) d(\text{smear}(\sigma_0)) \\ &= \int_{\Gamma \backslash G_+} \text{vol}(\text{Str}(\tilde{\phi}g\sigma_0)) d(\Gamma g) \\ &< (v_n - \epsilon) \text{vol}(\tilde{U}) + v_n (\text{vol}(M) - \text{vol}(\tilde{U})) \\ &= v_n \text{vol}(M) - \epsilon \text{vol}(\tilde{U}) \end{aligned}$$

$$\begin{aligned} \int_{\text{Str}(\phi_* \text{smear}(\rho\sigma_0))} dV_N &= \int_{\Gamma \backslash G_+} -\text{vol}(\text{Str}(\tilde{\phi}g\rho\sigma_0)) d(\Gamma g) \\ &\geq -v_n \text{vol}(M) \end{aligned}$$

$$\begin{aligned} \int_{\text{Str}(\phi_* \text{avg}(\sigma_0))} dV_N &< v_n \text{vol}(M) - \frac{\epsilon}{2} \text{vol}(\tilde{U}) \\ \text{vol}(\sigma_0) \text{vol}(N) &< v_n \text{vol}(M) - \frac{\epsilon}{2} \text{vol}(\tilde{U}) \end{aligned}$$

Since we may choose σ_0 to be arbitrarily close to a regular ideal simplex, with volume arbitrarily close to v_n , the above inequality contradicts the fact that $\text{vol}(N) = \text{vol}(M)$, which is guaranteed by the proportionality principle and Lemma 3.3. \square

Fix a particular regular ideal simplex $\Delta \subset \mathbb{H}^n$, with vertices p_0, \dots, p_n . Let G_Δ be the group of isometries generated by the reflections in the faces of Δ . It is an intuitive geometric fact that $G_\Delta \Delta = \bigcup_{g \in G_\Delta} g\Delta = \mathbb{H}^n$. In fact, this is true of any convex polyhedron; for a formal proof see Ratcliffe [Rat06], Theorem 7.1.1. An immediate consequence is the following.

Lemma 4.11. *The orbit union $\bigcup_i G_\Delta p_i$ is a dense subset of $\partial\mathbb{H}^n$.*

Proof. Suppose for contradiction that there is some open set $W \subset \partial\mathbb{H}^n$ disjoint from the orbits. W contains some disk $D \subset \partial\mathbb{H}^n$, whose boundary circle is the boundary for a hyperplane $H \subset \mathbb{H}^n$. Let H_+ be the half-space whose defined by H whose boundary

contains D . Half-spaces are convex, so the fact that $G_\Delta p_i \notin D$ for all i implies that $G_\Delta \Delta$ is disjoint from H_+ , which is a contradiction. \square

Now we return to our boundary map $\tilde{\phi}_\infty$. By Lemma 4.10, the points $\tilde{\phi}_\infty(p_i)$ are the vertices of a regular ideal simplex. Let ρ_i be the reflection in the i -th face of Δ , and ρ'_i be the reflection in the i -th face of $\text{Str}(\tilde{\phi}(\Delta))$. $\rho_i \Delta$ is a regular ideal simplex sharing the i -th face of Δ , so again $\tilde{\phi}_\infty$ must take its vertices to the vertices of some other regular ideal simplex, which can only be $\rho'_i \text{Str}(\tilde{\phi}(\Delta))$. That is,

$$\tilde{\phi}_\infty(\rho_i p_i) = \rho'_i \tilde{\phi}_\infty(p_i).$$

Proceeding by the same process, we see that the action of $\tilde{\phi}_\infty$ on the entire G_Δ orbits of the p_i is fully determined by the points $\tilde{\phi}_\infty(p_i)$. Since those G_Δ orbits are dense, this in fact fully determines the map $\tilde{\phi}_\infty$.

Independently, there is a unique isometry $f \in I(\mathbb{H}^n)$ which maps Δ onto $\text{Str}(\tilde{\phi}(\Delta))$, and thus the vertices p_i to $\tilde{\phi}_\infty(p_i)$. As an isometry, f shares with $\tilde{\phi}_\infty$ the property of permuting $(n+1)$ -tuples of points which define regular ideal simplices. It follows that $\tilde{\phi}_\infty$ is precisely the restriction of f to $\partial\mathbb{H}^n$.

By basic covering space theory, the lift $\tilde{\phi}$ satisfies an equivariance condition; for all $\gamma \in \Gamma$,

$$\tilde{\phi}\gamma = \phi_*(\gamma)\tilde{\phi},$$

where $\phi_* : \Gamma \rightarrow \Xi$ is the induced isomorphism on fundamental groups. Since isometries of \mathbb{H}^n are fully determined by their action on $\partial\mathbb{H}^n$, f must also be ϕ_* -equivariant. Therefore f descends to the desired isometry $\bar{f} : M \rightarrow N$. This concludes the proof of Mostow rigidity. \square

4.4. Further Research Directions.

Mostow rigidity gives a definitive answer to the question posed in the introduction for hyperbolic manifolds, proving that the geometry of such manifolds is uniquely determined by the topology. However, the details of this correspondence remain largely mysterious. For instance, there is no known algorithm for directly computing the volume of a hyperbolic manifold from a presentation of its fundamental group. Much ongoing research is devoted to finding connections between particular geometric and topological invariants.

A major example comes from knot theory. For many knots and links embedded in S^3 , the complement is a cusped, finite-volume hyperbolic 3-manifold. With a careful treatment of the cusps, our proof of Mostow rigidity extends to this case. The theorem then tells us that geometric invariants of the complement are in fact isotopy invariants of the knots. In fact, hyperbolic invariants, in particular the hyperbolic volume of the complement, have proven to be some of the best tools for distinguishing knots [HW98]. As knots are fundamentally combinatorial objects, there is much interest in finding direct connections between the volume of a knot and more readily accessible combinatorial invariants. Various upper bounds on the volume have been found in terms of the crossing number or the twist number of a knot, or in terms of the number and type of faces in the projection graph [Ada15]. One of the most famous open problems in the field, known as the volume conjecture, concerns a computation of the volume as the limit of an expression involving the colored Jones polynomial.

The examples are by no means limited to knot theory, or to the hyperbolic volume in particular. The simplicial volume alone has recently been used to put lower bounds on the number of maximally broken Morse trajectories in closed manifolds [Alp15], and on

the systolic volume of closed manifolds [Che15]. Over 40 years after its original proof, Mostow rigidity still hints at many more connections to be discovered between geometry and topology.

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