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The Grothendieck-Riemann-Roch Theorem

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1 Introduction

The classical Riemann-Roch theorem is a fundamental result in complex analysis and algebraic geometry. In its original form, developed by Bernhard Riemann and his student Gustav Roch in the mid-19th century, the theorem provided a connection between the analytic and topological properties of compact Riemann surfaces. This connection arises from relating data about the zeroes and poles of meromorphic functions on a surface to the genus of a surface. In the 1930s, Friedrich Karl Schmidt realized that this result could be proved in a purely algebraic context for smooth projective curves over arbitrary algebraically closed fields.

Theorem 1.1. *(Riemann-Roch) Let C be a smooth projective curve over an algebraically closed field k. Then for any divisor D on C, we have*

$$\ell(D) - \ell(K_C - D) = deg(D) - g(C) + 1, \tag{1.1}$$

where K_C is the canonical divisor of C and $g(C)$ is the genus of C, and for any divisor D, we denote by $\deg(D)$ *the degree of D and let $\ell(D) = \dim_k H^0(C, \mathscr{L}(C))$ be the dimension of the corresponding Riemann-Roch space.*

In the 1950s, Hirzebruch generalized this result to higher dimensional complex varieties and arbitrary vector bundles. Let E be a vector bundle on a smooth projective complex algebraic variety X, and denote by $\mathscr{E}$ the corresponding locally free sheaf. We define the *Euler-Poincaré characteristic* of E to be

$$\chi(X, E) = \sum_{i \geq 0} (-1)^i \dim_{\mathbb{C}} H^i(X, \mathscr{E}). \tag{1.2}$$

Note that $\dim_{\mathbb{C}} H^i(X, \mathscr{E}) < \infty$ for all i, since $\dim_{\mathbb{C}} H^i(X, \mathscr{E}) = 0$ for $i > \dim X$.

Theorem 1.2. *(Hirzebruch-Riemann-Roch) Let E be a vector bundle on a smooth projective complex algebraic variety X. Then*

$$\chi(X, E) = \int_X \operatorname{ch}(E) \cdot \operatorname{td}(T_X), \tag{1.3}$$

where $\operatorname{ch}(\cdot)$ *is the Chern character,* $\operatorname{td}(\cdot)$ *is the Todd class, and T_X is the tangent bundle to X (see Section 3.3 for relevant definitions).*

Essentially, Hirzebruch's theorem expresses a cohomological invariant of $\mathscr{E}$ in terms of appropriate characteristic classes of X.

Let $A^*(X)$ be the Chow ring of a quasi-projective non-singular variety X and let $K(X)$ be the ring induced by $K_0(X) \cong K^0(X)$. See §§2-4 for an explanation of these objects. Our goal will be the following theorem:

Theorem 1.3. *(Grothendieck-Riemann-Roch) Let $f : X \to Y$ be a proper morphism and let X and Y be quasi-projective non-singular varieties over an algebraically closed field k. Let $x \in K(X)$. Then the following diagram commutes:*

$$\begin{array}{ccc} K(X) & \xrightarrow{\operatorname{ch}(\cdot)\operatorname{td}(T_X)} & A^*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \\ \downarrow f_* & & \downarrow f_* \\ K(Y) & \xrightarrow[\operatorname{ch}(\cdot)\operatorname{td}(T_Y)]{} & A^*(Y) \otimes_{\mathbb{Z}} \mathbb{Q}\,. \end{array} \tag{1.4}$$

Equivalently,

$$f_*(\operatorname{ch}(x) \cdot \operatorname{td}(T_X)) = \operatorname{ch}(f_*(x)) \cdot \operatorname{td}(T_Y)). \tag{1.5}$$

The discovery of the Hirzebruch-Riemann-Roch theorem was a crucial moment for future generalizations of the classical theorem. Continuing in a purely algebraic setting, Grothendieck used this theorem to arrive at Grothendieck-Riemann-Roch. According to his philosophy of turning absolute statements about varieties into "relative" statements about morphisms, Grothendieck replaced the variety in Hirzebruch-Riemann-Roch with a morphism of schemes and the vector bundle with a coherent sheaf.

We will consider the broad strokes of Grothendieck's generalization. First, the base field $\mathbb{C}$ was replaced by an arbitrary base field; in this setting the analytic approach of Hirzebruch is not applicable. Second, the underlying cohomology ring was replaced with the Chow ring. Finally, all coherent sheaves were considered, not only the locally free category. Moreover, Grothendieck developed many new concepts along the way, e.g., a K-theory for schemes, and formulated new approaches to intersection theory and characteristic classes.

If we re-write Hirzebruch's theorem for a morphism, we get the following setup: let X be a compact complex variety and let $f : X \to \{\text{point}\}$. Then

$$\sum_{i \geq 0} (-1)^i \dim R^i f_* \mathscr{E} = f_*(\text{ch}(\mathscr{E}) \cdot \text{td}(T_X)), \tag{1.6}$$

where the morphism on the left-hand side is the direct image functor (global sections) and the morphism on the right-hand side is the integration map. Now take X to be a projective smooth variety over arbitrary field k with $f : X \to \text{Spec}(k)$. We want our new Riemann-Roch type theorem to be analogous to the above equation. Start by embedding X into $\mathbb{P}^n_k$ via a closed immersion $i : X \hookrightarrow \mathbb{P}^n_k$. Then, one must prove the above equation by replacing f with i and using facts about projective spaces. Grothendieck considered morphisms $f : X \to Y$ of smooth projective varieties. This can be reduced to the above case, since f can be factored as $f = p \circ i$ where $p : \mathbb{P}^n_Y \to Y$ and $i : X \hookrightarrow \mathbb{P}^n_Y$. We shall see that this factorization property is crucial to the proof of the theorem and further generalizations.

In what follows, all schemes will be connected, Noetherian, and quasi-projective over an algebraically closed field k, unless otherwise noted. In particular, all varieties will be quasi-projective over k. The purpose for these hypotheses will be discussed in the course of the essay.

1.1 Outline

We wish to prove the Grothendieck-Riemann-Roch theorem for non-singular quasi-projective varieties. This requires a great deal of preparatory theory: the construction of the Chow ring, a discussion of characteristic classes, and developing a K-theory for schemes. After dispensing with these requisite components, we will prove the theorem, and then briefly discuss some generalizations and applications.

Section 2: Let X be a smooth scheme. Here we will construct the Chow ring $A^*(X)$ of algebraic cycles on X modulo rational equivalence. This is the algebro-geometric analogue to singular cohomology in algebraic topology.

Section 3: Here we discuss characteristic classes, in particular Chern and Todd classes, with values taken in the Chow ring. These are our basic algebraic invariants of vector bundles. At this point we could state and prove the Hirzebruch-Riemann-Roch theorem.

Section 4: In this section, we introduce $K^0(X)$, the ring generated by equivalence classes of locally free sheaves, and $K_0(X)$, the group generated by equivalence classes of coherent sheaves, and show that for X a smooth algebraic variety, there exists a group isomorphism $K^0(X) \simeq K_0(X)$. This allows us to endow $K_0(X) = K(X)$ with the structure of a commutative ring. We then show that the Chern character, defined in the preceding section, yields a ring homomorphism $K(X) \to A^*(X)$.

Section 5: Here we complete the proof of Theorem 1.3.

Section 6: In this section, we discuss a general algebraic framework that is useful in the consideration of Riemann-Roch type theorems.

Section 7: In this section, we briefly discuss some consequences of the Grothendieck-Riemann-Roch theorem involving the γ-filtration.

2 Algebraic Cycles and the Construction of the Chow Ring

For any scheme X, we construct the associated Chow groups $A_*(X)$ and show that these have a commutative ring structure under the intersection product. A *variety* we will be a reduced and irreducible scheme, and a *subvariety* of a scheme will be a closed subscheme which is a variety. The classical reference for these results is [2]; a more complete exposition can be found there.

2.1 Algebraic Cycles

Definition 2.1. Let X be a scheme.The *group of k-cycles* on X, denoted by Z_kX, is the free abelian group generated by the k-dimensional subvarieties of X. For each k-dimensional subvariety $V \subset X$, we denote by $[V]$ the corresponding element of Z_kX. A *k-cycle* α on X is an element of Z_kX, i.e., a finite formal sum

$$\sum n_i[V_i] \tag{2.1}$$

where $n_i \in \mathbb{Z}$, and an *algebraic cycle* α on X is an element of the abelian group $\bigoplus_k Z_kX$.

The group of k-cycles Z_kX is rather large and cumbersome to work with. To remedy this, we define the notion of rational equivalence, which gives rise to a subgroup $\operatorname{Rat}_k X \subset Z_kX$, which is the subgroup of k-cycles that are rationally equivalent to zero. We recall that, for any $(k+1)$-dimensional subvariety W of X, and any $f \in R(W)^*$, the *divisor* of the function is

$$(f) = \sum \operatorname{ord}_V(f)[V], \tag{2.2}$$

where we sum over all codimension one subvarieties V of W. Here ord_V is the order of the function on $R(W)^*$, the non-zero elements of the field of rational functions on W, defined by the local ring $\mathscr{O}_{V,W}$.

Definition 2.2. An algebraic cycle α on X is *rationally equivalent to zero*, written as $\alpha \sim 0$, if there are subvarieties $V_1, \ldots, V_k$ of X and a rational function f_i for each V_i such that

$$\alpha = \sum_{i=1}^{k} (f_i) \tag{2.3}$$

in Z_kX. Furthermore, we say that two algebraic cycles α_1, α_2 are rationally equivalent if $\alpha_1 - \alpha_2$ is rationally equivalent to zero.

From this, we can give the definition of the Chow group.

Definition 2.3. The *Chow group* of k-cycles on X, denoted by $A_k(X)$, is the quotient of Z_kX b the subgroup $\operatorname{Rat}_k(X)$ of k-cycles rationally equivalent to 0. The direct sum

$$A_*(X) = \bigoplus_k A_k(X) \tag{2.4}$$

is called the *Chow group of* X. Furthermore, if $n = \dim(X)$, we let

$$A^i(X) = A^{n-i}(X) \tag{2.5}$$

for the classes of algebraic cycles of codimension i.

We have an alternate definition of rational equivalence to zero:

Definition 2.4. A cycle α in $Z_k(X)$ is rationally equivalent to zero iff there exist subvarieties $V_1, \ldots, V_k$ of $X \times \mathbb{P}^1$ such that the projection maps $\pi_i : V_i \to \mathbb{P}^1$ are dominant and

$$\alpha = \sum_{i=1}^{k} ([V_i(0)] - [V_i(\infty)]) \tag{2.6}$$

in $Z_k X$, where $[V_i(0)]$ and $[V_i(\infty)]$ are the scheme-theoretic fibers above 0 and ∞.

We can also give a definition of $A_*(X)$ that is of a more classical flavor: let X be a scheme and let $X_1, \ldots, X_k$ be irreducible components of X. Then each local ring $\mathscr{O}_{X_i,X}$ is zero-dimensional, i.e., Artinian, and the length $m_i = \ell_{\mathscr{O}_{X_i,X}}(\mathscr{O}_{X_i,X})$ as a module over itself is finite. We then define the *fundamental cycle* of X as

$$[X] = \sum_{i=1}^{k} m_i [X_i], \tag{2.7}$$

which is an element of $Z_*(X)$. However, by abuse of notation, we also write $[X]$ for its image in the Chow group. If $\dim X_i = k$ for all i, then $[X] \in Z_k X$. In this case $A_k X = Z_k X$ is the free abelian group on $[X_1], \ldots, [X_k]$.

Let's consider two simple examples:

Example 2.5. Since a scheme and its reduced scheme have the same subvarieties, the groups of cycles and rational equivalence classes are isomorphic:

$$A_k(X) \cong A_k(X_{\mathrm{red}}). \tag{2.8}$$

Example 2.6. If $n = \dim(X)$, then $A_n(X)$ is the free abelian group on the set of n-dimensional irreducible components of X. Furthermore, if X is a variety, then $A_{n-1}(X) \cong \mathrm{Cl}(X)$, the divisor class group of X.

2.2 Proper Pushforward of Cycles

Let $f : X \to Y$ be a proper morphism of schemes. For any subvariety V in X, the image $f(V)$ is a subvariety of Y with $\dim f(V) \leq \dim Y$. Thus, we can define

$$f_*[V] = \begin{cases} \deg(V/f(V)) \cdot [f(V)] & \dim(f(V)) = \dim(V) \\ 0 & \dim(f(V)) < \dim(Z), \end{cases} \tag{2.9}$$

where $\deg(V/f(V)) = [k(V) : k(f(V))]$ is the degree of the corresponding extension of fields of rational functions. Then we can linearly extend f_* to a functorial homomorphism of abelian groups: $f_* : Z_k X \to Z_k Y$.

Theorem 2.7. *Let $f : X \to Y$ be a proper morphism of schemes. Suppose that $\alpha \in Z_k X$ is rationally equivalent to zero. Then $f_*(\alpha)$ is rationally equivalent to zero on Y.*

Proof. Let us assume that $\alpha = (r)$, where r is a rational function on a $(k+1)$-dimensional subvariety $V \subset X$. We replace X by V and Y by $f(V)$, so we can take Y to be a variety and f surjective. Then we get our desired result from the following lemma:

Lemma 2.8. *Let $f : X \to Y$ be a proper, surjective morphism of varieties. Then*

$$f_*((r)) = \begin{cases} (N(r)) & \dim(Y) = \dim(X) \\ 0 & \dim(Y) < \dim(X), \end{cases} \tag{2.10}$$

where $N(r)$ is the norm of r. That is, for any field extension $K \hookrightarrow L$, every element $r \in L$ determines a K-linear endomorphism $m_r : L \to L$ via multiplication by r. Recall that the norm $N(r) = \det(m_r)$.

For a proof of this lemma, refer to [2], Proposition 1.4. □

From this theorem we can conclude that there is an induced homomorphism

$$f_* : A_k(X) \longrightarrow A_k(Y), \tag{2.11}$$

and that A_* is a covariant functor for proper morphisms.

We will need the following notion for the Hirzebruch-Riemann-Roch theorem:

Definition 2.9. Let $S = \mathrm{Spec}(k)$ and let $f : X \to S$ be the structure map. By Theorem 2.7, we get a map

$$f_* : A_0(X) \longrightarrow A_0(S) \simeq \mathbb{Z}, \tag{2.12}$$

i.e., the *degree* of an algebraic 0-cycle, which we extend to $A_*(X)$ by setting $f_*(A_k(X)) = 0$ for $k > 0$.Then we have the integration map

$$\int_X : A_*(X) \longrightarrow \mathbb{Z}. \tag{2.13}$$

Here the properness of f is crucial. Indeed, consider $f : \mathbb{A}^1 \to \mathrm{Spec}(k)$. Clearly, any element of the affine line is rationally equivalent to zero, but its pushforward is non-zero in

$$A_*(\mathrm{Spec}(k)) = A_0(\mathrm{Spec}(k)) \simeq \mathbb{Z}. \tag{2.14}$$

2.3 Pullback of Algebraic Cycles

Now that we have the pushforward map, we want to define the pullback on algebraic cycles, which will induce a map on the Chow groups. Consider a morphism of schemes $f : X \to Y$. In order to define the pullback map $f^* : A^*(Y) \to A^*(X)$, we require that f be a flat morphism of relative dimension n. Note that we switch from A_* to A^* because the pushforward preserves the dimension of an algebraic cycle, whereas the pullback preserves the codimension. We shall construct a few special cases of the pullback morphism in increasing generality.

For a flat morphism $f : X \to Y$ and a subvariety $V \subset Y$, we set

$$f^*[V] = [f^{-1}(V)], \tag{2.15}$$

where $f^{-1}(V)$ is the inverse image scheme, i.e., a subscheme of X with pure dimension $\dim(V) + n$, and $[f^{-1}(V)]$ is its associated algebraic cycle. This linearly extends to pullback homomorphisms

$$f^* : Z_k(Y) \longrightarrow Z_{k+n}(X). \tag{2.16}$$

We want a result analogous to Theorem 2.7 for the pullback operation in order to induce a map on the Chow groups.

Theorem 2.10. *Let $f : X \to Y$ be a flat morphism of relative dimension n, and α a cycle on Y that is rationally equivalent to zero. Then $f^*(\alpha)$ is rationally equivalent to zero as an algebraic $(i+n)$-cycle on X.*

Proof. (Sketch) Here we use the second definition of rational equivalence and reduce to the case where $\alpha = [V(0)] - [V(\infty)]$ with V a closed variety of $Y \times \mathbb{P}^1$. This amounts to proving a lemma that considers the restriction of a divisor to the irreducible components of X, but doing so on the level of algebraic cycles. See Theorem 1.7 of [2]. ☐

Thus, the pullback on the level of algebraic cycles induces a map on Chow groups:

$$f^* : A_k(Y) \to A_{k+n}(X). \tag{2.17}$$

Then A_* is a contravariant functor for flat morphisms.

It is useful to consider the commutativity of the proper pushforward and flat pullback on the level of algebraic cycles.

Theorem 2.11. *Let g be flat and f proper. If the Cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array} \tag{2.18}$$

commutes, then f' is proper and g' is flat. Furthermore, $f'_* g'^*(\alpha) = g^* f_*(\alpha)$ for all algebraic cycles $\alpha \in Z_* Y'$.

Proof. We may assume that X, Y are varieties, f is surjective, and $\alpha = [X]$. Let $f_*[X] = \deg(X/Y)[Y]$. We must show that $f'_*[X'] = \deg(X'/Y')[Y']$. We can do this locally: for fields L, K, let $\mathrm{Spec}(L) = X$ and $\mathrm{Spec}(K) = Y$, and for local rings A, B, let $\mathrm{Spec}(A) = Y'$ and $\mathrm{Spec}(B) = X'$ with A Artinian and $B = A \otimes_K L$. Then we simply apply the following lemma:

Lemma 2.12. *Let $A \to B$ be a local homomorphism of local rings. Let d be the degree of the residue field extension. A non-zero B-module M has finite length over A iff $d < \infty$ and M has finite length over B, in which case $\ell_A(M) = d \cdot \ell_B(M)$.*

Proof. We can reduce to the case where $M = B/\mathfrak{m}$ and $\mathfrak{m}$ is the maximal ideal of B. If $\mathfrak{p}$ is the maximal ideal of A, then

$$\ell_A(M) = \ell_{A/\mathfrak{p}}(B/\mathfrak{m}) = d, \tag{2.19}$$

since length and vector space dimension coincide on a field. ☐

We now have our desired result. ☐

Example 2.13. Let $f' : X' \to X$ be a finite, flat morphism. We know that each point $x \in X$ has an affine neighborhood U_x such that the coordinate ring of $f^{-1}(U_x)$ is a finitely generated free-module over the coordinate ring of U. If the rank of this module is d for all neighborhoods U_x, then f is of degree d. Then, for all subvarieties, $V \subset X$, $f_* f^*[X] = d[V]$ in $Z_*(X)$. That is, the composition of maps

$$A_*(X) \xrightarrow{f^*} A_*(X') \xrightarrow{f_*} A_*(X) \tag{2.20}$$

is simply multiplication by the degree d.

Finally, we close this subsection with a computationally useful proposition:

Proposition 2.14. *(Localization Sequence) Let Y be a closed subscheme of a scheme X. Set $U = X - Y$ with inclusion map $j : U \hookrightarrow X$. Then*

$$A_*(Y) \xrightarrow{i_*} A_*(X) \xrightarrow{j^*} A_*(U) \longrightarrow 0 \tag{2.21}$$

is an exact sequence.

Proof. Any subvariety $Z \subset U$ can be extended to a subvariety $Z' \subset X$, so we have an exact sequence

$$Z_k(Y) \xrightarrow{i_*} Z_k(X) \xrightarrow{j^*} Z_k(U) \longrightarrow 0\ . \tag{2.22}$$

Now suppose that, for any algebraic cycle $\alpha \in Z_k(X)$, $j^*\alpha$ is rationally equivalent to zero. Then

$$\alpha = \sum_{i=1} (r_i), \tag{2.23}$$

for a rational function r_i on subvarieties W_i of U. The function r_i corresponds to a rational function r_i' on W_i' and

$$j^* \left(\alpha - \sum_i (r_i') \right) = 0 \tag{2.24}$$

in $Z_k(U)$. Thus,

$$\alpha - \sum_i (r_i') = i_*\beta \tag{2.25}$$

for some cycle $\beta \in Z_k(Y)$, so we are done. □

2.4 Affine Bundles and Chow Groups

We now prove a proposition that allows us to compute the Chow groups over affine space. It will also be essential to our development of characteristic classes. We consider *affine bundles*, a generalization of the notion of vector bundles, in that there is no selection of linear structure on the fibers.

Definition 2.15. A scheme E with a morphism $\pi : E \to X$ is an affine bundle of rank n over X, if we can cover X by opens U_i and there are isomorphisms

$$\pi^{-1}(U_i) \cong U_i \times \mathbb{A}^n, \tag{2.26}$$

such that $\pi\mid_{\pi^{-1}(U_i)}$ is the projection from $U_i \times \mathbb{A}^n \to U_i$.

Proposition 2.16. *Let $\pi : E \to X$ be an affine bundle of rank n. Then the flat pullback*

$$\pi^* : A_k(X) \longrightarrow A_{k+n}(E) \tag{2.27}$$

is surjective for all k.

Proof. Take a closed subscheme $Y \subset X$ such that $U = X - Y$ is an affine open set over which E is trivial. Then we have a commutative diagram

$$\begin{array}{ccccccc} A_*(Y) & \longrightarrow & A_*(X) & \longrightarrow & A_*(U) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ A_*(\pi^{-1}(Y)) & \longrightarrow & A_*(E) & \longrightarrow & A_*(\pi^{-1}(U)) & \longrightarrow & 0 \end{array} \tag{2.28}$$

where the vertical arrows are flat pullbacks and the rows are exact by the preceding localization sequence. It suffices to prove the theorem for $E\mid_X$ and $E\mid_Y$. By Noetherian induction, we may take $X = U$ and assume that $E = X \times \mathbb{A}^n$. We consider $n = 1$ since $X \times \mathbb{A}^{n+1}$ is a trivial $\mathbb{A}^1$-bundle over $X \times \mathbb{A}^n$.

We must prove that, for any $(k+1)$-dimensional subvariety $V \subset E$, we can write

$$[V] = \sum_j n_j [Z_j \times \mathbb{A}^1] \tag{2.29}$$

as an element of $A_*(E)$, where the Z_j are k-dimensional subvarieties of X. We replace X by the closure of $\pi(V)$, so we may assume that X is a variety and π maps V dominantly to X so that $V = X \times \mathbb{A}^1$ or V is a divisor.

Assume that V is a divisor in E. Let A be the coordinate ring of X and $K = k(X)$ is the field of rational functions of X. Let $\mathfrak{p}$ be the prime ideal of A corresponding to V, so the dominance of π gives that $\mathfrak{p}\, K[t]$ is non-zero, where $K[t]$ is the coordinate ring of the generic fiber. Suppose that $r \in K[t]$ generates $\mathfrak{p}\, K[t]$ so

$$[V] - [(r)] = \sum_i n_i [V_i] \tag{2.30}$$

as an element of $A_*(E)$, where the V_i are $(k+1)$-dimensional subvarieties whose projections to X are not dominant. Thus, $V_i = \pi^{-1}(Z_i)$ for $Z_i = \pi(V_i) \subset X$. Therefore,

$$[V] = [(r)] + \sum_i \pi^*[Z_i], \tag{2.31}$$

and we have the surjectivity of π^*. ☐

Under suitable conditions, the flat pullback π^* is an isomorphism (see §3.2 for the definition of c_1, the first Chern class, and $\mathbb{P}(E)$):

Proposition 2.17. *Let E be an affine bundle of rank $r = e + 1$ on a scheme X with projection $\pi : E \to X$. Let $p : \mathbb{P}(E) \to X$ be the projectivization of E and $\mathscr{O}(1)$ the canonical line bundle on $\mathbb{P}(E)$. Then*

1. *The flat pullback is an isomorphism:*

$$\pi^* : A_{k-r}(X) \xrightarrow{\sim} A_k(E) \tag{2.32}$$

   for all k.

2. *Each element $\beta \in A_k(\mathbb{P}(E))$ is uniquely expressed as*

$$\beta = \sum_{i=0}^{e} c_1(\mathscr{O}(1))^i \cap p^*(\alpha_i) \tag{2.33}$$

   for all $\alpha_i \in A_{k-e+i}(X)$. Thus, the map

$$\theta_E : \bigoplus_{i=0}^{e} A_{k-e+i}(X) \longrightarrow A_k(\mathbb{P}(E)) \tag{2.34}$$

   is an isomorphism.

Proof. To prove that θ_E is surjective, we use the same inductive argument as above, in order to reduce E to the trivial case. By induction on e, it suffices to prove that θ_F is surjective when θ_E is surjective and $F = E \oplus 1$, where 1 is the trivial line bundle $X \times \mathbb{A}^1 \to X$.

We note that $\mathbb{P}(E \oplus 1) = \mathbb{P}(E) \sqcup E$ where $i : \mathbb{P}(E) \to \mathbb{P}(E \oplus 1)$ is a closed embedding and $j : E \to \mathbb{P}(E \oplus 1)$ is an open embedding. Let $q : \mathbb{P}(E \oplus 1) \to X$ be the projection. We modify our localization sequence:

$$\begin{array}{ccccccc} A_k(\mathbb{P}(E)) & \xrightarrow{i_*} & A_k(\mathbb{P}(E\oplus 1)) & \xrightarrow{j^*} & A_k(E) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle q^*} & \nearrow{\scriptstyle \pi^*} & & & \\ & & A_{k-r}(X) & & & & \end{array} \tag{2.35}$$

We know that the top row is exact and that π^* is surjective. Thus, given any $\beta \in A_k(\mathbb{P}(E\oplus 1))$ there is an $\alpha \in A_{k-r}(X)$ such that $j^*\beta = \pi^*\alpha$. Hence, $\beta - q^*\alpha \in \ker j^*$. By the exactness of the top row, and since we inductively assume that θ_E is surjective, we have that

$$\beta - q^*\alpha = i_*\left(\sum_{i=0}^{e} c_1(\mathscr{O}_E(1))^i \cap p^*(\alpha_i)\right). \tag{2.36}$$

We note that $i^*(\mathscr{O}_F(1)) = \mathscr{O}_E(1)$. Then by the projection formula (see below):

$$\sum_{i=0}^{e} c_1(\mathscr{O}_F(1))^i \cap i_*p^*(\alpha_i) = \sum_{i=0}^{e} c_1(\mathscr{O}_F(1))^i \cdot c_1(\mathscr{O}_F(1)) \cdot q^*(\alpha). \tag{2.37}$$

This last step, namely that $i_*p^*(\alpha) = c_1(\mathscr{O}_F(1))\cdot q^*(\alpha)$ holds because both sides are the effect of the pullback of α to $\mathbb{P}(E\oplus 1)$ and then intersecting with the divisor $\mathbb{P}(E\oplus 1)$. That is, we have the following:

Lemma 2.18. *For all $\alpha \in A_*(X)$*

$$c_1(\mathscr{O}_F(1)) \cap q^*\alpha = i_*p^*(\alpha). \tag{2.38}$$

Proof. It suffices to prove this for $\alpha = [V]$ where V is a subvariety of X. We know that $\mathscr{O}_F(1)$ has a section vanishing on $\mathbb{P}(E)$, so the equality

$$c_1(\mathscr{O}_F(1)) \cap [q^{-1}V] = [p^{-1}V] \tag{2.39}$$

follows from the definition of the Chern class. □

Hence, β lies in the image of θ, proving surjectivity. To prove the uniqueness of (2), suppose there is a non-trivial relation

$$\beta = \sum_{i=0}^{e} c_1(\mathscr{O}(1))^i \cap p^*(\alpha_i) = 0. \tag{2.40}$$

Let ℓ be the largest integer such that $\alpha_\ell \neq 0$. Then

$$p_*(c_1(\mathscr{O}(1))^{e-\ell} \cap \beta) = \alpha_\ell, \tag{2.41}$$

which is a contradiction. Hence, θ is an isomorphism.

Finally, to prove the injectivity of π^* we again let $F = E \oplus 1$. If $\pi^*\alpha = 0$ with $\alpha \neq 0$, then $j^*q^*\alpha = 0$, so

$$q^*\alpha = i_*\left(\sum_{i=0}^{e} c_1(\mathscr{O}_E(1))^i \cap p^*\alpha_i\right) = \sum_{i=0}^{e} c_1(\mathscr{O}_F(1))^{i+1} \cap q^*\alpha_i \tag{2.42}$$

by the previous lemma. However, this is a contradiction as we proved the uniqueness of (2) for $E \oplus 1$. □

Corollary 2.19. *The Chow groups of any open subset $U \subset \mathbb{A}^n$ are*

$$A_i(U) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & \textit{otherwise.} \end{cases} \tag{2.43}$$

Proof. Apply the above proposition to $\mathbb{A}^n \to \{\text{point}\}$ to deduce the result for $U = \mathbb{A}^n$. Then, using the localization sequence, $A_*(\mathbb{A}^n)$ surjects onto $A_*(U)$. □

2.5 The Chow Ring

From Proposition 2.17, we can define an important intersection operation: the Gysin homomorphisms. Using these, we will construct the Chow ring. Refer to [2] for full details.

Definition 2.20. Let s be the zero section of a vector bundle E. Then $s : X \to E$ with $\pi \circ s = \mathrm{id}_X$. The *Gysin homomorphisms*

$$s^* : A_k E \longrightarrow A_{k-r} X \tag{2.44}$$

are defined by

$$s^*(\beta) = (\pi^*)^{-1}(\beta), \tag{2.45}$$

where β is defined in Proposition 2.17 and $r = \mathrm{rank}(E)$.

Note that, for any subvariety $V \subset E$, or k-cycle β on E, regardless of how it intersects the zero section, there is always a well-defined class $s^*(\beta)$ in $A_{k-r}(X)$. By the surjectivity of π^*, the homomorphism s^* is determined by $s^*[\pi^{-1}(V)] = [V]$ for all $V \subset X$, and the fact that s^* preserves rational equivalence. Using this, we will define a more general Gysin homomorphism.

Let $i : X \to Y$ be a regular embedding of codimension d, and let $f : Y' \to Y$ be a morphism. Consider the fiber square

$$\begin{array}{ccc} X' & \xrightarrow{j} & Y' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[i]{} & Y \end{array} \tag{2.46}$$

and define homomorphisms

$$i^! : Z_k Y' \to A_{k-d} X' \tag{2.47}$$

explicitly by

$$i^!\left(\sum n_i [V_i]\right) = \sum n_i X \cdot V_i, \tag{2.48}$$

where $X \cdot V_i$ is the intersection product (see [2] 6.1). Now, we want $i^!$ to pass to rational equivalence, so we give a slightly modified definition:

Definition 2.21. (Refined Gysin Homomorphism) We define $i^!$ as the composition

$$Z_k Y' \xrightarrow{\sigma} Z_k C' \longrightarrow A_k N \xrightarrow{s^*} A_{k-d} X' \tag{2.49}$$

where $C' = C_{X'} Y'$ is a closed subcone of $N = g^* N_X Y$, $\sigma : Z_k Y \to Z_k C$ is the specialization homomorphism defined by $\sigma[V] = [C_{V \cap X} V]$ for any k-dimensional subvariety V of Y, and s^* is the Gysin homomorphism for zero-sections defined above. The specialization homomorphism σ passes to rational equivalence ([2] 5.2), so $i^!$ does as well.

The induced homomorphisms

$$i^! : A_k Y' \to A_{k-d} X' \tag{2.50}$$

are called *refined Gysin homomorphisms.* When $Y' = Y$ and $f = \mathrm{id}_Y$ these are called simply *Gysin homomorphisms* and are denoted $i^* : A_k Y \to A_{k-d} X$.

Let X be a smooth scheme of dimension n. Then the diagonal embedding $\Delta : X \to X \times X$ is a regular embedding of codimension n. This gives a product operation:

$$A_p X \otimes A_q X \longrightarrow A_{p+q}(X \times X) \xrightarrow{\Delta^*} A_{p+q-n} X \ , \tag{2.51}$$

where Δ^* is the Gysin homomorphism defined above. Taking the upper index, we get a global intersection product:

$$A^p(X) \otimes A^q(X) \longrightarrow A^{p+q}(X). \tag{2.52}$$

Then $A^*(X)$ is a commutative graded ring, called the *Chow Ring*.

Now consider $f : X \to Y$ a morphism of smooth schemes. Let

$$\Gamma_f := \{(x, f(x)) : x \in X\} \subset X \times Y, \tag{2.53}$$

be the *graph morphism* of f, which is a regular embedding. For $x \in A_* X, y \in A^* Y$, we define

$$x \cdot y = \Gamma_f^*(x \times y) \in A_* X. \tag{2.54}$$

Then $A_* X$ is a graded module over $A^* Y$. Finally, since both X and Y are smooth, we can define a general pullback

$$f^* : A^* Y \longrightarrow A^* X \tag{2.55}$$

by $f^*(y) = [X] \cdot y$.

We should now make note of the *projection formula* mentioned in the previous section. For a morphism $f : X \to Y$, we can take both $A^*(X)$ and $A^*(Y)$ to be $A^*(Y)$-modules. Then the proper pushforward $f_* : A^*(X) \to A_*(Y)$ is a homomorphism of $A^*(Y)$-modules. Namely, we have a projection formula for Chow groups:

$$f_*(f^*(y) \cdot x) = y \cdot f_*(x) \tag{2.56}$$

for all $x \in A^*(X)$ and $y \in A^*(Y)$.

3 Characteristic Classes on the Chow Ring

Here we discuss the basic theory of characteristic classes taking values in the Chow ring. This is typically done over the category of smooth manifolds; however, we would like a more functorial description, and follow [3] in our development. In our construction of characteristic classes we assume all schemes are smooth.

A *characteristic class* is an element $a(E) \in A_*(X)$ associated to a given vector bundle E on a scheme X. This association is compatible with the pullback operation, but not with the pushforward. Indeed, the Grothendieck-Riemann-Roch theorem studies the failure of commutativity of a particular characteristic class and the (proper) pushforward on the Chow ring.

3.1 Invertible Sheaves

Characteristic classes may be formulated in sheaf-theoretic terms, as we have a functorial correspondence between vector bundles and locally free sheaves. We recall some useful definitions and demonstrate this correspondence.

Definition 3.1. An $\mathscr{O}_X$-module $\mathscr{F}$ is *free* if it is isomorphic to a direct sum of copies of $\mathscr{O}_X$. Also, $\mathscr{F}$ is *locally free* if X can be covered by open sets U for which $\mathscr{F}|_U$ is a free $\mathscr{O}_X|_U$-module. Then we define the *rank* of $\mathscr{F}$ on such an open set U to be the number of necessary copies of the structure sheaf. If X is connected, then the rank of a locally free sheaf is constant. A locally free sheaf of rank 1 is said to be an *invertible sheaf.*

We now define a *vector bundle* and show that this is equivalent to the notion of a locally free sheaf of finite rank.

Definition 3.2. Let Y be a scheme. Then a *(geometric) vector bundle* of rank n over Y is a scheme X and a morphism $f : X \to Y$, along with the data:

1. An open covering $\{U_i\}_{i\in I}$ of Y.
2. Isomorphisms $\varphi_i : f^{-1}(U_i) \to \mathbb{A}^n_{U_i}$ such that for all i, j and any open subset $\operatorname{Spec}(A) \subseteq U_i \cap U_j$, the automorphisms $\varphi = \varphi_j \circ \varphi_i^{-1}$ of $\operatorname{Spec} A[x_1, \dots, x_n]$ is A-linear.

Proposition 3.3. *There is a one-to-one correspondence between isomorphism classes of locally free sheaves on Y of rank n and isomorphism classes of vector bundles of rank n over Y.*

Proof. See exercise II.5.17 of [6]. Roughly speaking, if $\mathscr{F}$ is a locally free sheaf of rank n, we can select a set of n generators $x_1, \dots, x_n$ for the $\mathscr{O}_X(U)$-modules of $\mathscr{F}(U)$. They span an n-dimensional affine space $A[x_1, \dots, x_n]$ over U, where A is the coordinate ring over U. We change to another set of generators over another open subset and write down the transition functions. Thus, we associate to $\mathscr{F}$ a vector bundle structure. Conversely, if E is a vector bundle on X, locally we have $\mathscr{F}|_U \cong U \times \mathbb{A}^n$ with a basis $x_1, \dots, x_n$ of $\mathbb{A}^n$ over U. Then we can associate to $\mathscr{F}|_U$ an $\mathscr{O}_X(U)$-module of rank n using $x_1, \dots, x_n$ as generators. □

Now we may use vector bundles and locally free sheaves interchangeably. Furthermore, we consider only locally free sheaves of finite rank. Finally, the notions of tensor product, direct sum, exterior product, and Hom agree with one another on vector bundles and locally free sheaves.

3.2 Chern Classes

Let X be a smooth scheme. Let $\operatorname{Pic}(X)$ be the group of invertible sheaves on X and let $\operatorname{Cl}(X) = A^1(X)$ be the divisor class group. Every divisor D on X determines (up to isomorphism) an invertible sheaf $\mathscr{O}_X(D)$ and every invertible sheaf is of this type. This induces an isomorphism $\operatorname{Cl}(X) \xrightarrow{\sim} \operatorname{Pic}(X)$.

Definition 3.4. For every $\mathscr{L} \in \mathrm{Pic}(X)$, we define the *first Chern class* of $\mathscr{L}$ to be $c_1(\mathscr{L}) = [D]$, where $[D] \in \mathrm{Cl}(X)$ is such that $\mathscr{O}_X(D) = \mathscr{L}$ in $\mathrm{Pic}(X)$. Clearly, the homomorphism $c_1 : \mathrm{Pic}(X) \to \mathrm{Cl}(X)$ is the inverse of $\mathrm{Cl}(X) \xrightarrow{\sim} \mathrm{Pic}(X)$. Recall that this isomorphism $\mathrm{Cl}(X) \simeq \mathrm{Pic}(X)$ holds only when X is factorial, e.g., if X is smooth.

Definition 3.5. Let E be a vector bundle with a corresponding locally free sheaf $\mathscr{E}$. The associated *projective bundle* (the projectivization of E) is defined as

$$\mathbb{P}(E) = \mathbf{Proj}(S(\mathscr{E})) \to X, \tag{3.1}$$

where the symmetric algebra is degree-graded. There is a canonical line bundle $\mathscr{O}(1) = \mathscr{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$, whose fiber over (x, p) is the line in E_x corresponding to the point p.

Let $p : E \to X$ be a vector bundle of rank n and let $\pi : \mathbb{P}(E) \to X$ be the associated projective bundle. Note that the pullback $\pi^* : A(X) \to A(\mathbb{P}(E))$ makes $A(\mathbb{P}(E))$ into an $A(X)$-module.

Definition 3.6. Let $u = c_1(\mathscr{O}_{\mathbb{P}(E)}(1))$. There are unique elements $c_i \in A^i(X)$ with $1 \leq i \leq n$ such that

$$u^n - c_1(E) \cdot u^{n-1} + c_2(E) \cdot u^{n-2} - \cdots + (-1)^n c_n(E) = 0. \tag{3.2}$$

We call the $c_i(E) \in A^i(X)$ the *Chern classes* of E. Furthermore, we define the *total Chern class* of E to be

$$c(E) = 1 + c_1(E) + \cdots + c_n(E) \in A^*(X). \tag{3.3}$$

Definition 3.7. We define the *Chern polynomial* $c_t(E) \in A(X)[t]$ to be

$$c_t(E) = 1 + c_1(E)t + \cdots + c_n(E)t^n. \tag{3.4}$$

In [5], Grothendieck developed a theory of Chern classes that assigns to a vector bundle E on a smooth scheme X, a Chern class $c_i(E) \in A^i(X)$ for all $i \geq 0$. The $c_i(E)$ have the following properties:

Theorem 3.8. *Let E be a vector bundle. Then the Chern classes satisfy:*

1. $c_0(E) = 1$.
2. *For an invertible sheaf $\mathscr{O}_X(D)$, which is the line bundle corresponding to a divisor D, we have that $c_1(\mathscr{O}_X(D)) = [D]$.*
3. *(Pullback) For a morphism $f : X \to Y$ of smooth quasi-projective varieties $f^*(c_i(E)) = c_i(f^*(E))$ for all i.*
4. *(Whitney Sum) If we have an exact sequence of vector bundles*

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0\ , \tag{3.5}$$

$c_t(E) = c_t(E')c_t(E'')$ *in* $A(X)[t]$.

5. $c_i(E) = 0$ *for all $i > n$ where $n = rank(E)$.*

A key ingredient in the proof of this theorem is the Splitting Principle, which will be of great utility later on:

Theorem 3.9. *(Splitting Principle) Let E be a vector bundle of rank n over a smooth scheme X. Then there is a smooth scheme Y and a flat morphism $f : Y \to X$ such that*

$$f^* : A^*(X) \longrightarrow A^*(Y) \tag{3.6}$$

is a split monomorphism of abelian groups and there is a filtration

$$0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = f^*E, \tag{3.7}$$

where each E_i/E_{i-1} is a line bundle.

Before proving this theorem, we require a lemma:

Lemma 3.10. *Let E be a vector bundle of rank n on a smooth scheme X. Let $p : \mathbb{P}(E) \to X$ be the corresponding projective bundle. Then for any $\alpha \in A_k(X)$,*

$$p_*(c_1(\mathscr{O}(1))^{n-1} \cdot p^*\alpha) = \alpha. \tag{3.8}$$

Proof. We first check compatibility with proper morphisms. Let $f : Y \to X$ be a proper morphism and let π be the projection of $f^*(\mathbb{P}(E))$, then

$$f_*(\pi_*(c_1(\mathscr{O}_{f^*\,\mathbb{P}(E)}(1))^{n-1} \cdot \pi^*\alpha) = p_*(c_1(\mathscr{O}_{\mathbb{P}(E)}(1))^{n-1} \cdot p^* f_*(\alpha)). \tag{3.9}$$

Thus, we may consider $\alpha = [X]$ and $p^*(\alpha) = [\mathbb{P}(E)]$. Then we get

$$p_*(c_1(\mathscr{O}(1))^{n-1} \cdot [\mathbb{P}(E)]) = q[X], \tag{3.10}$$

for some integer q. Now we prove a similar equality for flat morphisms. Let $f : Y \to X$ be flat. Then by naturality:

$$\pi_*(c_1(\mathscr{O}_{f^*\,\mathbb{P}(E)}(1))^{n-1} \cdot \pi^* f^*(\alpha)) = f^* p_*(c_1(\mathscr{O}_{\mathbb{P}(E)}(1))^{n-1} \cdot \pi^*(\alpha)). \tag{3.11}$$

We can show this locally since any local inclusion is flat. Take E to be the trivial bundle, i.e., $\mathbb{P}(E) = X \times \mathbb{P}^{n-1}$. Then there is a section of $\mathscr{O}(1)$ with zero section $X \times \mathbb{P}^{n-2}$, so

$$c_1(\mathscr{O}(1)) \cdot [X \times \mathbb{P}^{n-1}] = [X \times \mathbb{P}^{n-2}]. \tag{3.12}$$

Iterate $n-1$ times to get that $q = 1$ and we are done. □

We now prove the Splitting Principle:

Proof. Induct on the rank n of E. The base case is immediate. As before, take $\mathbb{P}(E)$ to be the projective bundle associated to E. By the above lemma, π^* is injective on the Chow ring. We have an exact sequence

$$0 \longrightarrow \mathscr{O}(-1) \longrightarrow \pi^*E \longrightarrow L \longrightarrow 0\ , \tag{3.13}$$

since the pullback bundle π^*E contains $\mathscr{O}(-1)$ as a rank 1 sub-bundle. Now consider $L' = \pi'^*L$ where $\pi' : \mathbb{P}(L) \to \mathbb{P}(E)$ is the projection. Then the pullback of $\pi \circ \pi'$ is injective and there is a filtration $L'_{n-1} \subset L'_n = L'$ with line bundle quotients. □

The main application of the Splitting Principle is in the following factorization of the total Chern class:

$$c(E) = \prod_{i=1}^{n}(1 + c_1(L_i)), \tag{3.14}$$

where the L_i are the quotient line bundles from the proof of the theorem. This is merely a "formal" factorization in that we really factor the pullback of E by f. From this factorization, we may write

$$c(E) = \prod_{i=1}^{n}(1 + \alpha_i), \tag{3.15}$$

where $\alpha_i = c_1(L_i)$ are called the *Chern roots* of the splitting. Furthermore, note that $c_i(E)$ is the i-th elementary symmetric polynomial in the Chern roots $\alpha_1, \ldots, \alpha_n$. In particular, $c_1(E) = \alpha_1 + \cdots + \alpha_n$ and $c_n(E) = \alpha_1 \cdots \alpha_n$. Thus, any symmetric polynomial in the Chern roots can be expressed in terms of Chern classes and gives a well-defined invariant of E.

Example 3.11. If the vector bundle E has a filtration with quotients L_i, then its dual E^* has a filtration with quotients L^*_{n-i}. Then, if we have Chern roots $\alpha_1, \dots, \alpha_n$ for E, we will have Chern roots $-\alpha_1, \dots, \alpha_n$ for E^*. Thus,

$$c_i(E^*) = (-1)^i c_i(E). \tag{3.16}$$

Example 3.12. Again taking $\alpha_1, \dots, \alpha_n$ to be the Chern roots of E, then

$$c_t(\wedge^p E) = \prod_{i_1 < \dots < i_p} (1 + (\alpha_{i_1} + \cdots + \alpha_{i_n})t). \tag{3.17}$$

For the first Chern class:

$$c_1(\wedge^n E) = c_1(E). \tag{3.18}$$

Note that any exact sequence of vector bundles

$$0 \longrightarrow L \longrightarrow E \longrightarrow E' \longrightarrow 0 \tag{3.19}$$

where L is a line bundle induces a short exact sequence

$$0 \longrightarrow \bigwedge^{p-1} E' \otimes L \longrightarrow \bigwedge^p E \longrightarrow \bigwedge^p E' \longrightarrow 0\ . \tag{3.20}$$

3.3 The Chern and Todd Characters

Definition 3.13. The *Chern character* $\mathrm{ch}(E)$ of a vector bundle E is an element of $A^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ defined as

$$\mathrm{ch}(E) = \sum_{i=1}^{n} \exp(\alpha_i) \tag{3.21}$$

with $\exp(x) = \sum_{n=0}^{\infty} x^n/n!$ and the α_i are Chern roots of E.

Here we have used $A^*(X)_{\mathbb{Q}}$ in order to make sense of the $1/n!$ factors in the exponential series. Note that

$$\mathrm{ch}(E \otimes E') = \mathrm{ch}(E) \cdot \mathrm{ch}(E'), \tag{3.22}$$

since the Chern roots of a tensor product add. Now consider a short exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0\ . \tag{3.23}$$

Theorem 3.8(4) tells us that if the Chern roots of E' are $\alpha_1, \dots, \alpha_n$ and the Chern roots of E'' are $\beta_1, \dots, \beta_k$, then the Chern roots of E are $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_k$. Thus, we get

$$\mathrm{ch}(E) = \sum_{i=1}^{n} \exp(\alpha_i) + \sum_{j=1}^{k} \exp(\beta_j) = \mathrm{ch}(E') + \mathrm{ch}(E''). \tag{3.24}$$

Since $\mathrm{ch}(E)$ is symmetric in the Chern roots, we can express it in terms of Chern classes:

$$\mathrm{ch}(E) = r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + c_3) + \frac{1}{24}(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4) + \cdots, \tag{3.25}$$

where $c_i = c_i(E)$ and $r = \mathrm{rank}(E)$. Here we have grouped the terms by equal degree in $A^*(X)_{\mathbb{Q}}$.

Now we define the *Todd class* of E:

Definition 3.14. The Todd class $\mathrm{td}(E)$ of a vector bundle E is given by

$$\mathrm{td}(E) = \prod_{i=1}^{r} Q(\alpha_i) \tag{3.26}$$

where $Q(x)$ is the power series

$$Q(x) = \frac{x}{1-e^{-x}} = 1 + \frac{1}{2}x + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k}{(2k)!}x^{2k}. \tag{3.27}$$

The B_k are the k-th Bernoulli numbers and $\alpha_1, \ldots, \alpha_r$ are the Chern roots of E.

As with the Chern characters, we can express the Todd class in terms of Chern classes:

$$\mathrm{td}(E) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + 3c_2^2 + c_1c_3 - c_4) + \cdots . \tag{3.28}$$

The relation,

$$\mathrm{td}(E) = \prod_{i=1}^{n} Q(\alpha_i) \cdot \prod_{j=1}^{k} Q(\beta_j) = \mathrm{td}(E') \cdot \mathrm{td}(E''), \tag{3.29}$$

holds using the same short exact sequence of vector bundles.

4 K-theory of Schemes

In this section we introduce the K-groups $K^0(X)$ and $K_0(X)$ of a scheme X. These are abelian groups constructed from the categories of locally free and coherent sheaves, respectively. Let $\mathfrak{Loc}(X)$ be the category of locally free sheaves, let $\mathfrak{Coh}(X)$ be the category of coherent sheaves, and finally let $\mathfrak{Mod}(X)$ be the category of $\mathscr{O}_X$-modules. The primary goal of this section will be the proof of the fact that $K^0(X)$ and $K_0(X)$ are equivalent on a non-singular quasi-projective variety.

Definition 4.1. Let A be a ring and M an A-module. We define the sheaf associated to M on Spec(A), denoted by $\widetilde{M}$ in the following way: for each prime ideal $\mathfrak{p} \subseteq A$, let $M_\mathfrak{p}$ be the localization of M at $\mathfrak{p}$. Then, for any open set $U \subseteq \operatorname{Spec}(A)$, we define

$$\widetilde{M}(U) := \left\{ s : U \longrightarrow \bigcup_{\mathfrak{p}\in U} M_\mathfrak{p} : s(\mathfrak{p}) \in M_\mathfrak{p}, \text{ locally } s \text{ is a fraction } m/f \text{ with } m \in M \text{ and } f \in A \right\}. \quad (4.1)$$

Proposition 4.2. *Let A be a ring and let M be an A-module. Let $\widetilde{M}$ be the sheaf associated to M on $X = \operatorname{Spec}(A)$ and let $\mathscr{O}_X$ be the structure sheaf of X. Then*

1. *$\widetilde{M}$ is an $\mathscr{O}_X$-module;*
2. *For each $\mathfrak{p} \in X$, the stalks are localizations $(\widetilde{M})_\mathfrak{p} \cong M_\mathfrak{p}$.*

Next, we recall the definition of a quasi-coherent sheaf:

Definition 4.3. Let X be a scheme. Then a sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$ is *quasi-coherent* if X can be covered by opens $U_i = \operatorname{Spec}(A_i)$, such that for all i there is an A_i-module M_i with $\mathscr{F}|_{U_i} \cong \widetilde{M_i}$. We write $\mathfrak{Qcoh}$ for the category of quasi-coherent sheaves. Furthermore, $\mathscr{F}$ is called *coherent* if each M_i is a finitely generated A_i-module.

Since we will construct $K_0(X)$ from the category $\mathfrak{Coh}(X)$, we take all of our schemes to be Noetherian, as coherent sheaves are well-behaved on Noetherian schemes only. We collect some useful properties of quasi-coherent and coherent sheaves. See [6] II.5 for proofs.

Proposition 4.4. *Let $X = \operatorname{Spec}(A)$ be an affine scheme, $f \in A$ and let $D(f)$ be the distinguished open determined by f. For a quasi-coherent sheaf $\mathscr{F}$ on X:*

1. *If $s \in \Gamma(X, \mathscr{F})$ is a global section of $\mathscr{F}$ whose restriction to $D(f)$ is zero, then there is some $n > 0$, such that $f^n s = 0$.*
2. *Given a section $t \in \mathscr{F}(D(f))$ of $\mathscr{F}$ over the open set $D(f)$, then for some $n > 0$, $f^n t$ extends to a global section of $\mathscr{F}$ over X.*

Proposition 4.5. *Let X be an affine scheme and let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence of $\mathscr{O}_X$-modules with $\mathscr{F}'$ quasi-coherent. Then*

$$0 \longrightarrow \Gamma(X, \mathscr{F}') \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{F}'') \longrightarrow 0 \quad (4.2)$$

is exact.

Proposition 4.6. *Let X and Y be schemes and $f : X \to Y$ a morphism of scheme. Then the following hold:*

1. *The kernel, cokernel, and image of any morphism of quasi-coherent sheaves are quasi-coherent. Assuming X is Noetherian, the result holds for coherent sheaves.*

2. *If $\mathscr{F}$ is quasi-coherent sheaf of $\mathscr{O}_Y$-modules, then $f^* \mathscr{F}$ is a quasi-coherent sheaf of $\mathscr{O}_X$-modules.*

3. *If X, Y are Noetherian, and if $\mathscr{F}$ is coherent, then $f^* \mathscr{F}$ is coherent.*

4. *Assume either X Noetherian or f quasi-compact and separated. Then if $\mathscr{F}$ is a quasi-coherent sheaf of $\mathscr{O}_X$-modules, $f_* \mathscr{F}$ is a quasi-coherent sheaf of $\mathscr{O}_Y$-modules.*

Proposition 4.7. *Let X be Noetherian and let U be an open subset of X. Let $\mathscr{F}$ be a coherent sheaf on U. Then there is a coherent sheaf $\mathscr{G}$ such that $\mathscr{G} \restriction_U \cong \mathscr{F}$. Moreover, if there is a coherent sheaf $\mathscr{G}$ on X with $\mathscr{F} \subset \mathscr{G} \restriction_U$, then there is a coherent sheaf $\mathscr{F}'$ on X which extends $\mathscr{F}$ such that $\mathscr{F}' \subset \mathscr{G}$.*

We now recall the following result of Serre, which guarantees the existence of locally free resolutions for coherent sheaves on quasi-projective schemes.

Theorem 4.8. *(Serre) Let $X \subset \mathbb{P}^n$ be a quasi-projective scheme over a Noetherian ring A. Then any coherent sheaf $\mathscr{F}$ on X can be written as a quotient of a sheaf $\mathscr{E}$, where $\mathscr{E}$ is a finite direct sum of twisted structure sheaves $\mathscr{O}(n_i)$ with $n_i \in \mathbb{Z}$. In particular, we have an exact sequence $\mathscr{E} \to \mathscr{F} \to 0$ with $\mathscr{E}$ locally free.*

Proof. Consider $X = \mathbb{P}^n$ with $\mathscr{F}$ a coherent sheaf on X. We have an embedding $i : X \hookrightarrow \mathbb{P}^n$. Then there is an extension $\mathscr{F}'$ of $\mathscr{F}$ to $\overline{X}$, the closure of X in $\mathbb{P}^n$. Then $\mathscr{F}'$ is the quotient of a direct sum of twisted Serre sheaves, which is locally free on $\overline{X}$ by the $\mathbb{P}^n$ case. Hence, the restriction of $\mathscr{F}'$ to X is as well.

Now suppose that $X = \mathbb{P}^n$ and write $\mathscr{O}_X = \mathscr{O}_{\mathbb{P}^n}$. If we can generate $\mathscr{F}(n) = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}(n)$, for some $n \in \mathbb{Z}$, by a finite number of global sections, then we are done. This would give

$$\bigoplus_{i=1}^N \mathscr{O}_X \longrightarrow \mathscr{F}(n) \longrightarrow 0 \,. \tag{4.3}$$

Tensoring with $\mathscr{O}(-n)$ gives the result. Now we need to show the existence of these global sections. Cover X with the usual affines U_i which have coordinate rings

$$A_i = \left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right]. \tag{4.4}$$

We have assumed that $\mathscr{F}$ is coherent, so $\mathscr{F}_{U_i} = \widetilde{M_i}$ where the M_i are A_i-modules. For each i we have a finite number of elements $s_{ij} \in M_i$ that generate the module. Then for some n, there is an $x_i^n s_{ij}$ that extends to a global section t_{ij} of $\mathscr{F}(n)$ by the above proposition. On each open affine U_i, $\mathscr{F}(n)$ corresponds to some $\widetilde{M'_i}$ and $x_i^n : \mathscr{F} \to \mathscr{F}(n)$ induces an isomorphism $M_i \xrightarrow{\sim} M_i'$, and hence the global sections $t_{ij} \in \Gamma(X, \mathscr{F}(n))$ generate $\mathscr{F}(n)$ everywhere. $\square$

If X is a Noetherian scheme such that the last statement in the above theorem holds, we say that X has *enough locally frees.*

4.1 Grothendieck Groups

Definition 4.9. Let $\mathscr{C}$ be a full additive subcategory of an abelian category $\mathscr{A}$. Recall that a category is *additive* if:

1. For any objects $X, Y \in \mathscr{C}$, then $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ has an abelian group structure.

2. For any objects $X, Y, Z \in \mathscr{C}$, the map $\mathrm{Hom}_{\mathscr{C}}(X, Y) \times \mathrm{Hom}_{\mathscr{C}}(Y, Z) \to \mathrm{Hom}_{\mathscr{C}}(X, Z)$ is bilinear.

3. $\mathscr{C}$ has a zero object.

4. For any $X, Y \in \mathscr{C}$, $X \times Y$ is in $\mathscr{C}$.

Furthermore, a category $\mathscr{C}$ is *abelian* if: every morphism in $\mathscr{C}$ has a kernel and cokernel, and every monomorphism is a kernel and every epimorphism is a cokernel.

We note that $\mathfrak{Coh}(X)$ is abelian on any Noetherian scheme X, although $\mathfrak{Loc}(X)$ is not. However, $\mathfrak{Loc}(X)$ is an exact category, i.e., a full additive subcategory $\mathscr{C}$ of an abelian category $\mathscr{A}$ such that if $0 \to A' \to A \to A'' \to 0$ is a short exact sequence in $\mathscr{A}$ with $A', A'' \in \mathscr{C}$, then $A \in \mathscr{C}$.

Definition 4.10. Let $\mathrm{Ob}(\mathscr{C})$ be the class of objects of $\mathscr{C}$, and let $Q = \mathrm{Ob}(\mathscr{C})/\sim$ be the set of isomorphism classes. Let $F(\mathscr{C})$ be the free abelian group on Q, namely, any element of $F(\mathscr{C})$ can be written as a finite formal sum

$$\sum n_X[X] \tag{4.5}$$

with $[X]$ an isomorphism class of $X \in \mathrm{Ob}(\mathscr{C})$ and $n_X \in \mathbb{Z}$. Furthermore, to any sequence S

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \tag{4.6}$$

in $\mathscr{C}$ which is exact in $\mathscr{A}$, we associate the element $G(S)$, generated by the symbol $[B]-[A]-[C]$ in $F(\mathscr{C})$. Let $H(\mathscr{C})$ be the subgroup generated by the elements of $G(S)$. Then the *Grothendieck group*, denoted by $K(\mathscr{C})$, is the quotient

$$K(\mathscr{C}) := F(\mathscr{C})/H(\mathscr{C}). \tag{4.7}$$

Let G be an abelian group. Then a function $\varphi : Q \to G$ on the set of isomorphism classes of objects of $\mathscr{C}$ is called *additive*, if for every short exact sequence $0 \to A \to B \to C \to 0$, the relation

$$\varphi(B) = \varphi(A) + \varphi(C) \tag{4.8}$$

holds. The Grothendieck group $K(\mathscr{C})$ satisfies the following universal property:

Proposition 4.11. *Let $\varphi : Q \to G$ be an additive function. Then there is a unique abelian group homomorphism $\widetilde{\varphi} : K(\mathscr{C}) \to G$, such that $\varphi = \widetilde{\varphi} \circ \pi$, where $\pi : Q \to K(\mathscr{C})$ is the canonical projection.*

Let X be a Noetherian scheme. The two Grothendieck groups that will be of interest to us are:

$$K^0(X) = K(\mathfrak{Loc}(X)) = K(\mathfrak{Vec}(X)), \tag{4.9}$$

where $\mathfrak{Vec}(X)$ is the category of vector bundles on X, and

$$K_0(X) = K(\mathfrak{Coh}(X)). \tag{4.10}$$

Note that by construction, we have the relation $[\mathscr{F}_1 \oplus \mathscr{F}_2] = [\mathscr{F}_1] + [\mathscr{F}_2]$ in both groups.

4.2 The Grothendieck Group of Coherent Sheaves

We establish some important properties of $K_0(X)$ for a Noetherian scheme X. We should note that the category $\mathfrak{Coh}(X)$ is a full abelian subcategory in the category of $\mathscr{O}_X$-modules. If $X = \mathrm{Spec}(A)$ is affine, then the global section functor gives an equivalence of categories from $\mathfrak{Coh}(X)$ to the category of finitely generated A-modules (see [6] II.5). Indeed, for a ring A, we write $K_0(A) = K_0(\mathrm{Spec}(A))$ and $K_0(A)$ is the Grothendieck group associated to the category of finitely generated A-modules.

Let $f : X \to Y$ be proper morphism of schemes. Recall that for $f : X \to Y$ a continuous map of topological spaces and any sheaf $\mathscr{F}$ on X, the direct image sheaf $f_*\mathscr{F}$ on Y is defined by

$$(f_*\mathscr{F})(V) = \mathscr{F}(f^{-1}(V)) \tag{4.11}$$

for any open set $V \subseteq Y$. Then f_* is a functor from the category of sheaves on X to the category of sheaves on Y.

Example 4.12. For a closed immersion $f : X \to Y$, the direct image coincides with the extension by zero of a sheaf. In this case, f_* is exact. For a field k and $f : X \to \mathrm{Spec}(k)$, the pushforward is precisely $\Gamma(X, \cdot)$, which is left exact only. Its right derived functors in the category of sheaves on X are the cohomology functors $H^i(X, \cdot)$.

Definition 4.13. For $f : X \to Y$ a continuous map of topological spaces and $i \geq 0$, the *higher direct image functors* are defined to be the right derived functors

$$R^i f_* : \mathfrak{Mod}(X) \longrightarrow \mathfrak{Mod}(Y), \tag{4.12}$$

of the direct image functor f_*.

Proposition 4.14. *Let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Then $R^i f_*(\mathscr{F})$ is the sheaf associated to the presheaf*

$$U \mapsto H^i\left(f^{-1}(U), \mathscr{F}\restriction_{f^{-1}(U)}\right) \tag{4.13}$$

on Y.

Proof. (Sketch) We note that $\mathscr{H}^i(X, \mathscr{F})$ is the sheaf associated to the presheaf above. This is a δ-functor (see [5] III.i) from the category of sheaves of abelian groups on X to the same category on Y. When $i = 0$, $f_* \mathscr{F} = \mathscr{H}^0(X, \mathscr{F})$. Furthermore, for some injective object $\mathscr{I}$, the right derived functors vanish when $i > 0$.

Now $\mathscr{I}\restriction_{f^{-1}(U)}$ for each open U is an injective object of $\mathfrak{Ab}(f^{-1}(U))$. Thus, $\mathscr{H}^i(X, \mathscr{I}) = 0$ for $i > 0$. Then by Theorem 1.3A in III.i of [5], we have a unique isomorphism $R^i f_*(\cdot) \simeq \mathscr{H}^i(X, \cdot)$ of δ-functors. □

Proposition 4.15. *Again with X, Y topological spaces, and $\mathscr{F}$ any quasi-coherent sheaf on X, let $Y =$ Spec(A) be affine. Then*

$$R^i f_*(\mathscr{F}) \cong \widetilde{H^i(X, \mathscr{F})}. \tag{4.14}$$

Then $R^i f_*(\mathscr{F})$ is quasi-coherent, even without Y being affine.

Proof. Quasi-coherence comes easily by looking at Y locally. Since the functor $\widetilde{(\cdot)}$ from A-modules to $\mathfrak{Mod}(Y)$ is exact, both functors $R^i f_*(\cdot)$ and $\widetilde{H^i(\cdot)}$ are δ-functors $\mathfrak{Qcoh}(X) \to \mathfrak{Mod}(Y)$. One can embed $\mathscr{F}$ in a flabby quasi-coherent sheaf, so both functors are effaceable for $i > 0$. Thus, we have a unique δ-functor isomorphism between them. See [6] III.8 for full details. □

Seeing that $R^i f_*(\mathscr{F})$ is coherent is a little more difficult (and this is not true for an arbitrary morphism), so we let $f : X \to Y$ be projective rather than proper. From Chow's lemma, we can see that projective morphisms are reasonably similar to proper morphisms ([6] II.iv Exercise 10). Now, in order to get coherence, we must show that $H^i(X, \mathscr{F})$ is a finitely generated A-module when $f : X \to \mathrm{Spec}(A)$ is projective. This is a well-known theorem of Serre. We have a closed immersion $i : X \hookrightarrow \mathbb{P}^n_A$ for some integer n. If $\mathscr{F}$ is coherent on X, then $i_* \mathscr{F}$ is coherent on $\mathbb{P}^n_A$ and the cohomology coincides. Thus, we may reduce to the case $X = \mathbb{P}^n_A$. In this case, we use Čech cohomology computations to show that $H^i(X, \mathscr{F})$ is finitely generated for sheaves $\mathscr{O}_X(r)$, $r \in \mathbb{Z}$. The same is true for direct sums of such sheaves.

For an arbitrary coherent sheaf $\mathscr{F}$ on X, use descending induction on i. For $i > n$, $H^i(X, \mathscr{F})$ vanishes, since X can be covered by $n + 1$ open affines. Now consider the short exact sequence

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{E} \longrightarrow \mathscr{F} \longrightarrow 0\,, \tag{4.15}$$

where $\mathscr{E}$ is the direct sum of sheaves $\mathscr{O}(r_i)$ and $\mathscr{K}$ is the kernel, which is also coherent. This gives a long exact sequence of A-modules

$$\cdots \longrightarrow H^i(X, \mathscr{E}) \longrightarrow H^i(X, \mathscr{F}) \longrightarrow H^{i+1}(X, \mathscr{K}) \longrightarrow \cdots \tag{4.16}$$

Now, since A is Noetherian, we need only prove the finite generation of the left and right modules above to get that $H^i(X, \mathscr{F})$ is finitely generated. Clearly, the left is finitely generated because $\mathscr{E}$ is the direct sum of sheaves $\mathscr{O}_X(r_i)$. The right is finitely generated by the inductive hypothesis.

Of course, we want $f : X \to Y$ to be a proper morphism. Since we assume that X, Y are quasi-projective, let $i : X \hookrightarrow \mathbb{P}^n$ be a closed immersion. Let $\pi : \mathbb{P}^n \times Y \to Y$ be the projection onto the first factor. Then for any proper morphism $f : X \to Y$, we have the factorization

$$X \xrightarrow{(i,f)} \mathbb{P}^n \times Y \xrightarrow{\pi} Y \tag{4.17}$$

which gives f as a closed immersion into $\mathbb{P}^n \times Y$ followed by projection. Thus, f is a projective morphism.

Corollary 4.16. *Let $f : X \to Y$ be a proper morphism of quasi-projective schemes. Then f is projective and $R^i f_*(\mathscr{F}) \in \mathfrak{Coh}(Y)$ with $\mathscr{F}$ coherent on X.*

We will always consider quasi-projective varieties, so we are always in the above situation. From the preceding corollary, we get a number of essential facts. First, given any coherent sheaf $\mathscr{F}$ on X, the element $[R^i f_*(\mathscr{F})]$ is well-defined in $K_0(Y)$. Indeed, for any proper morphism $f : X \to Y$ and $\mathscr{F} \in \mathfrak{Coh}(X)$, we define the pushforward on K_0 as the homomorphism

$$f_* : K_0(X) \longrightarrow K_0(Y), \qquad [\mathscr{F}] \mapsto \sum_{i \geq 0} (-1)^i [R^i f_*(\mathscr{F})]. \tag{4.18}$$

This map is well-defined by Proposition 4.15, as there are only finitely many non-zero cohomology groups. Furthermore, it is induced by the long exact sequence for right derived functors. However, we are not quite finished, as we want to establish the naturality of f_* on K_0, which is rather difficult. We appeal to the Grothendieck Spectral Sequence to get this result. This spectral sequence will also be required in our proof of the Grothendieck-Riemann-Roch theorem for a closed immersion. This spectral sequence is also used in versions of the theorem where X, Y are not quasi-projective to establish that $R^i f_*(\mathscr{F})$ is coherent, and thus the existence of f_* on K_0.

Recall the definition of a cohomological spectral sequence:

Definition 4.17. A *spectral sequence* in an abelian category $\mathscr{A}$ consists of the following data:

1. A family $\{E_r^{p,q}\}$ of objects of $\mathscr{A}$ with $r \geq a$ and beginning with E_a;
2. Morphisms $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ that satisfy $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$ where

$$E_{r+1}^{p,q} \cong \frac{\ker(d_r^{p,q})}{\operatorname{im}(d_r^{p-r,q-r+1})}. \tag{4.19}$$

A cohomological spectral sequence is said to be *bounded* if there are only finitely many nonzero terms in each total degree in E_a^{**}. Namely, for each p and q there is an r_0 such that $E_r^{p,q} = E_{r+1}^{p,q}$ for all $r \geq r_0$. We denote the stable value of the terms $E_r^{p,q}$ by $E_\infty^{p,q}$ and say that the bounded spectral sequence *converges* to H^* if we have a family of objects H^n of $\mathscr{A}$, each having a finite filtration

$$0 = F^t H^n \subseteq \cdots F^{p+1} H^n \subseteq F^p H^n \cdots \subseteq F^s H^n = H^n \tag{4.20}$$

such that

$$E_\infty^{p,q} \cong F^p H^{p+q} / F^{p+1} H^{p+q}. \tag{4.21}$$

We write $E_a^{p,q} \Rightarrow H^{p+q}$ for bounded convergence.

Theorem 4.18. *(Grothendieck Spectral Sequence) Let $F : \mathscr{B} \to \mathscr{C}$ and $G : \mathscr{A} \to \mathscr{B}$ be left exact functors of abelian categories, where $\mathscr{A}, \mathscr{B}$ have enough injectives. Suppose that G sends injective objects of $\mathscr{A}$ to F-acyclic objects of $\mathscr{B}$. Then for each $A \in ob(\mathscr{A})$, there is a first quadrant cohomological spectral sequence with*

$$E_2^{p,q} = (R^p F)(R^q(G))(A) \Rightarrow R^{p+q}(GF)(A). \tag{4.22}$$

Proof. See 5.7 and 5.8 of [9] for a proof using hyper-derived functors. □

From this theorem we have the naturality of f_* on K_0. We conclude this section by noting that K_0 is a covariant functor from the category of Noetherian, finite dimensional schemes with proper morphisms to the category of abelian groups.

4.3 The Grothendieck Group of Locally Free Sheaves

Let $f : X \to Y$ be a morphisms of Noetherian schemes. For a coherent $\mathscr{O}_X$-module $\mathscr{G}$ on Y, the pullback $f^*\mathscr{G}$ is also coherent. This is also true for local freeness. However, f^* is only right exact in general, so it does not descend to a pullback on K_0.

Lemma 4.19. *The mappings $\mathscr{E} \mapsto f^*\mathscr{E}$ and $f \mapsto f^*$ form an exact functor $\mathfrak{Loc}(Y) \to \mathfrak{Loc}(X)$.*

Thus, K^0 is a well-defined contravariant functor to the category of abelian groups. For $[\mathscr{E}] \in K^0(Y)$, we write $f^*[\mathscr{E}]$ for $[f^*\mathscr{E}] \in K^0(X)$. Finally, naturality $(g \circ f)^* = f^* \circ g^*$ is immediate.

It is important to note that we have a ring structure defined on $K^0(X)$ by $\otimes_{\mathscr{O}_X}$.

Lemma 4.20. *Tensor product of locally free sheaves makes $K^0(X)$ a ring.*

Proof. The tensor product gives a ring structure to the free abelian group $\mathbb{Z}[Q(\mathfrak{Loc}(X))]$ on the elements of $Q(\mathfrak{Loc}(X))$. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of locally free $\mathscr{O}_X$-modules. Then the subgroup

$$\mathscr{H} = \{[B] - [A] - [C]\} \tag{4.23}$$

is an ideal in $\mathbb{Z}[Q(\mathfrak{Loc}(X))]$, since locally free $\mathscr{O}_X$-modules are flat. Hence, the quotient,

$$K^0(X) = \mathbb{Z}[Q(\mathfrak{Loc}(X))]/\mathscr{H} \tag{4.24}$$

is a ring also. □

This also defines $K_0(X)$ as a $K^0(X)$-module under the same operation. Finally, note that the tensor product commutes with pullback of sheaves, so that f^* is a contravariant functor to the category of rings.

Remark 4.21. For any vector bundle E on X, there is a locally constant map: $\text{rk} : X \to \mathbb{Z}$ sending $x \in X$ to the rank of E_x. This defines a homomorphism $K^0(X) \to H^0(X, \mathbb{Z})$. For a connected scheme, $\ker(\text{rk} : K^0(X) \to \mathbb{Z})$ is the beginning of the γ-filtration for $K^0(X)$. See §7.

Remark 4.22. All line bundles are invertible as elements of $K^0(X)$. That is, $[\mathscr{L}]^{-1} = [\mathscr{H}om(\mathscr{L}, \mathscr{O}_X)]$. Recall that an invertible sheaf on a ringed space X is a locally free $\mathscr{O}_X$-module of rank 1 and that the Picard group $\text{Pic}(X)$ is the group of isomorphism classes of invertible sheaves on X under the operation $\otimes_{\mathscr{O}_X}$. It can be shown that $\text{Pic}(X) \cong H^1(X, \mathscr{O}_X^*)$

4.4 The Equality of $K_0(X)$ and $K^0(X)$

We have an obvious homomorphism, called the *Cartan homomorphism*,

$$\delta : K^0(X) \longrightarrow K_0(X) \tag{4.25}$$

induced by the embedding $\mathfrak{Loc}(X) \to \mathfrak{Coh}(X)$.

We also have a K-theory analogue to the projection formula on Chow groups:

Proposition 4.23. *(Projection Formula) Let $f : X \to Y$ be a proper morphism of schemes. Then*

$$f_*(f^*(y) \cdot x) = y \cdot f_*(x), \tag{4.26}$$

where $x \in K_0(X)$ and $y \in K^0(X)$.

Proof. It suffices to prove this for $x = [\mathscr{F}]$ and $y = [\mathscr{G}]$, where $\mathscr{F}$ is a coherent sheaf on X and $\mathscr{G}$ is a locally free sheaf on Y. Note there is a natural isomorphism of coherent sheaves

$$f_*(f^*(\mathscr{G} \otimes \mathscr{F})) = \mathscr{G} \otimes f_*(\mathscr{F}). \tag{4.27}$$

This is a local statement, so we let $Y = \operatorname{Spec}(A)$ and $\mathscr{G} = \mathscr{O}_Y^r$. Then, by the definition of f^*, we have that $f^* \mathscr{F} = \mathscr{O}_X^r$. This gives the above. Then, for any locally free sheaf $\mathscr{G}$, the functor

$$\mathscr{E} \mapsto f_*(f^*(\mathscr{G}) \otimes \mathscr{E}) \tag{4.28}$$

is left exact and its right derived functors coincide with those of $\mathscr{E} \mapsto \mathscr{G} \otimes f_* \mathscr{E}$. Since $\mathscr{G}$ is flat, we get

$$R^i f_*(\mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{G} = R^i f_*(\mathscr{F} \otimes_{\mathscr{O}_X} f^* \mathscr{G}), \tag{4.29}$$

which implies the projection formula. Now let's explicitly compute the formula:

$$f_*(f^*[\mathscr{G}] \cdot [\mathscr{F}]) = \sum_{i \geq 0} (-1)^i [R^i f_*(f^*(\mathscr{G}) \cdot \mathscr{F})] = \sum_{i \geq 0} (-1)^i [\mathscr{G} \otimes R^i f_*(\mathscr{F})] \tag{4.30}$$

$$= \sum_{i \geq 0} (-1)^i [\mathscr{G}] \cdot [R^i f_*(\mathscr{F})] = [\mathscr{G}] \cdot f_*[\mathscr{F}]. \tag{4.31}$$

The second equality follows from (4.29) and the third equality comes from the definition of the $K^0(X)$-module structure. □

With the Cartan homomorphism and the Projection formula in hand, we are now prepared to prove the equality of $K^0(X)$ and $K_0(X)$ for X smooth and quasi-projective.

Lemma 4.24. *Let $\mathscr{F} \in \mathfrak{Coh}(X)$. Select a finite locally free resolution $0 \to \mathscr{G}_n \to \cdots \to \mathscr{G}_0 \to \mathscr{F} \to 0$ and consider*

$$\xi(\mathscr{F}) = \sum_{i=0}^{n} (-1)^i [\mathscr{G}_i] \in K^0(X), \tag{4.32}$$

which is independent of the resolution and depends on $\mathscr{F}$.

Proof. Let $[\mathscr{G}_.] \in K^0(X)$ denote the value of $\xi(\mathscr{F})$ obtained by using the resolution $\mathscr{G}_.$. Then let $\mathscr{G}_. \to \mathscr{F}$ and $\mathscr{G}'_. \to \mathscr{F}$ be two finite locally free resolutions of $\mathscr{F}$.

By Lemma A.6, there is a third resolution $\mathscr{G}''_.$ and surjections $\mathscr{G}''_. \to \mathscr{G}_.$, $\mathscr{G}''_. \to \mathscr{G}'_.$, which give the identity map on H_0. By Lemma A.3, these have length at most $n = \dim(X)$. Thus, we must show that $[\mathscr{G}''_.] = [\mathscr{G}_.]$.

Let $\mathscr{G}_{1,\cdot} = \ker(\mathscr{G}''_\cdot \to \mathscr{G}_\cdot)$. By Lemma A.1, this consists of only locally free sheaves. In the induced long exact homology sequence, $H_i(\mathscr{G}''_\cdot) = 0$ and $H_i(\mathscr{G}_\cdot) = 0$, which cause $H_i(\mathscr{G}_{1,\cdot})$ to vanish for all i. Thus, $[\mathscr{G}_{1,\cdot}] = 0$ and

$$[\mathscr{G}''_\cdot] = [\mathscr{G}_\cdot] + [\mathscr{G}_{1,\cdot}] = [\mathscr{G}_\cdot]. \tag{4.33}$$

□

Thus, we have a well-defined map $\xi : Q(\mathfrak{Coh}(X)) \to Q(\mathfrak{Loc}(X))$, given by $[\mathscr{F}] \mapsto \sum_{i=0}^{n}(-1)^i[\mathscr{G}_i]$. This map also respects exact sequences:

Lemma 4.25. *For an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of coherent sheaves, we have that $\xi(\mathscr{F}) = \xi(\mathscr{F}') + \xi(\mathscr{F}'')$.*

This lemma tells us that ξ descends to a homomorphism $\xi : K_0(X) \to K^0(X)$. We now prove that this is the inverse to the Cartan homomorphism.

Theorem 4.26. *For a non-singular quasi-projective variety X, the Cartan homomorphism $\delta : K^0(X) \to K_0(X)$ is an isomorphism.*

Proof. Take $\mathscr{F} \in \mathfrak{Coh}(X)$ and $\mathscr{G} \in \mathfrak{Loc}(X)$. We need only prove

$$(\xi \circ \delta)[\mathscr{G}] = [\mathscr{G}], \qquad (\delta \circ \xi)[\mathscr{F}] = [\mathscr{F}]. \tag{4.34}$$

Consider the resolution $0 \to \mathscr{G} \to \mathscr{G} \to 0$ of $\mathscr{G}$. Then we have that

$$(\xi \circ \delta)[\mathscr{G}] = \xi[\mathscr{G}] = [\mathscr{G}]. \tag{4.35}$$

Similarly, we choose a resolution $0 \to \mathscr{G}_0 \to \cdots \to \mathscr{G}_0\, \mathscr{F} \to 0$ to see that

$$(\delta \circ \xi)[\mathscr{F}] = \delta\left([\mathscr{G}_0] - \ldots + (-1)^n[\mathscr{G}_n]\right) = [\mathscr{G}_0] - \ldots + (-1)^n[\mathscr{G}_n] = [\mathscr{F}]. \tag{4.36}$$

□

Therefore, since $K^0(X)$ has a ring structure given by the tensor product, $K_0(X)$ inherits this structure by the Cartan homomorphism. Thus, when X is a non-singular quasi-projective variety, $K_0(X) = K^0(X)$, which we denote by $K(X)$.

For $\mathscr{F}, \mathscr{G} \in \mathfrak{Coh}(X)$, we can now explicitly compute the product $[\mathscr{F}] \cdot [\mathscr{G}] \in K(X)$. Taking locally free (that is, projective) resolutions and taking the derived functor of the tensor product, we have

$$[\mathscr{F}] \cdot [\mathscr{G}] = \sum_{i \geq 0}(-1)^i[\mathrm{Tor}_i^{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})]. \tag{4.37}$$

Some authors, in particular Borel-Serre, take this to be definitional.

We note that $K(X)$ has what is called a λ*-ring structure*. In essence, exterior powers descend to $K(X)$. For a locally free sheaf $\mathscr{G}$ on X, we define

$$\lambda^i[\mathscr{G}] = [\wedge^i \mathscr{G}] \tag{4.38}$$

and linearly extend this to $K(X)$. We also define a map

$$\lambda : K(X) \to K(X)[[t]] \tag{4.39}$$

by writing

$$\lambda_t[\mathscr{G}] = 1 + \sum_{i>1} \lambda^i[\mathscr{G}] \cdot t^i. \tag{4.40}$$

In particular,

$$\lambda_{-1}[\mathscr{G}] = 1 - [\mathscr{G}]t + [\wedge^2 \mathscr{G}]t^2 - \cdots + (-1)^r[\wedge^r \mathscr{G}]t^r, \tag{4.41}$$

where $r = \mathrm{rank}(\mathscr{G})$.

5 Grothendieck-Riemann-Roch

We are now ready to prove the Grothendieck-Riemann-Roch theorem for non-singular quasi-projective varieties over an algebraically closed field k; our exposition follows the original proof in [1]. We have introduced $K(X)$ as a more tractable alternative to the free abelian group of isomorphism classes of either coherent $\mathscr{O}_X$-modules or locally free $\mathscr{O}_X$-modules, where both $\mathrm{ch} : K(X) \to A^*(X)_{\mathbb{Q}}$ and $\mathrm{td} : K(X) \to A^*(X)_{\mathbb{Q}}$ are well-defined. Our theorem considers the naturality of the Chern character ring homomorphism with respect to the proper pushforward, expressed as:

Theorem 5.1. *(Grothendieck-Riemann-Roch) Let $f : X \to Y$ be a proper morphism, X and Y are quasi-projective non-singular varieties over an algebraically closed field k. Let $x \in K(X)$. Then the following diagram commutes:*

$$\begin{array}{ccc} K(X) & \xrightarrow{\mathrm{ch}(\cdot)\,\mathrm{td}(T_X)} & A^*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ K(Y) & \xrightarrow[\mathrm{ch}(\cdot)\,\mathrm{td}(T_Y)]{} & A^*(Y) \otimes_{\mathbb{Z}} \mathbb{Q}\,. \end{array} \tag{5.1}$$

Equivalently,

$$f_*(\mathrm{ch}(x) \cdot \mathrm{td}(T_X)) = \mathrm{ch}(f_*(x)) \cdot \mathrm{td}(T_Y)). \tag{5.2}$$

The proof of the theorem will be broken into two stages:

1. Establish that $f : X \times \mathbb{P}^n \to X$ is projection onto the first factor.
2. Show that $f : Y \to X$ is an immersion onto a closed subvariety.

Lemma 5.2. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be proper morphisms. Let $x \in K(X)$ and set $y = f_*(x)$. Then:*

1. *If GRR is true for (f, x) and for (g, y) respectively, then it is true for (fg, x).*
2. *If GRR is true for (g, y) and for (fg, x) and if g_* is injective (on the Chow ring), then GRR is true for (f, x).*

Proof. 1. We have that

$$(fg)_*(\mathrm{ch}(x)\,\mathrm{td}(T_X)) = g_*(\mathrm{ch}(f_*(x))\,\mathrm{td}(T_Y)) \tag{5.3}$$
$$= \mathrm{ch}(g_*(f_*(x)))\,\mathrm{td}(T_X), \tag{5.4}$$

where the first equality is by the GRR for (f, x) and the second equality by GRR for (g, y).

2. Setting

$$u = f_*(\mathrm{ch}(x)\,\mathrm{td}(T_X)), \qquad v = \mathrm{ch}(y)\,\mathrm{td}(T_Y) \tag{5.5}$$

we wish to prove that $u = v$. However, it suffices to prove that $g_*(u) = g_*(v)$. We see that

$$g_*(u) = (fg)_*(\mathrm{ch}(x)\,\mathrm{td}(T_X)) \tag{5.6}$$
$$= \mathrm{ch}((fg)_*(x)\,\mathrm{td}(T_X)) \tag{5.7}$$
$$= \mathrm{ch}(g_*(y)\,\mathrm{td}(T_X)) \tag{5.8}$$
$$= g_*(\mathrm{ch}(y)\,\mathrm{td}(T_Y)) \tag{5.9}$$
$$= g_*(v), \tag{5.10}$$

where the second equality holds by GRR for (fg, x) and the fourth equality holds by GRR for (g, y).

□

Taking two quasi-projective varieties X and Y, we write their product as $X \times Y$. Consider the projections $X \times Y \to X$ and $X \times Y \to Y$. These define homomorphisms $K(X) \to K(X \times Y)$ and $K(Y) \to K(X \times Y)$. These induced pullback homomorphisms give another map

$$K(X) \otimes K(Y) \to K(X \times Y) \tag{5.11}$$

whose image consists of the tensor product $x \times y$ of two elements $x \in X$ and $y \in Y$. We now prove a statement analogous to Lemma 5.2 for this map.

Lemma 5.3. *Let $f : X \to Y$ and $f' : X' \to Y'$ be proper morphisms, and let $x \in K(X)$ and $x' \in K(X')$. If GRR is true for (f, x) and for (f', x'), then it is true for $(f \times f', x \otimes x')$, where $f \times f' : X \times X' \to Y \times Y'$.*

Proof. The proof of this lemma is entirely analogous to the preceding one, but study how the pushforward maps (K-pushforward and Chow pushforward) and the Chern character behave with the product map. Thus, we have three statements to verify:

1. $(f \times f')_*(x \otimes x') = f_*(x) \otimes f_*(x')$
2. $(f \times f')_*(\eta \otimes \eta') = f_*(\eta) \otimes f'_*(\eta')$
3. $\mathrm{ch}(x \otimes x') = \mathrm{ch}(x) \otimes \mathrm{ch}(x')$,

where $\eta \in A^*(X)$ and $\eta' \in A^*(X')$.

To prove (1) we invoke Proposition 4.15 and use the Künneth formula for coherent sheaf cohomology. Then the statement follows by the definitions of the product maps. To prove (2), take a cycle $\alpha \in Z$ and a cycle $\alpha' \in Z'$. If either cycle is rationally equivalent to zero, then their product is rationally equivalent to zero on the product variety. The third statement follows directly from the multiplicativity of the Chern character on $\mathfrak{Loc}(X)$. Thus, by these equalities, we have

$$(f \times f')_*(\mathrm{ch}(x \otimes x')\,\mathrm{td}(T_{X \times X'})) = \mathrm{ch}((f \times f')_*(x \otimes x'))\,\mathrm{td}(T_{Y \times Y'}). \tag{5.12}$$

□

By Lemma (5.3), in order to prove that $f : X \times \mathbb{P}^n \to X$ is the projection onto the first factor, we must show that

1. The homomorphism $K(X) \otimes K(\mathbb{P}^n) \to K(X \times \mathbb{P}^n)$ is surjective.
2. GRR is true for the case where $\mathbb{P}^n$ is simply a point. That is, Hirzebruch-Riemann-Roch holds:

$$\chi(\mathbb{P}^n, \mathscr{F}) = \int_{\mathbb{P}^n} \mathrm{ch}(\mathscr{F})\,\mathrm{td}(T_{\mathbb{P}^n}) \tag{5.13}$$

for any coherent sheaf $\mathscr{F}$.

In order to prove the surjectivity of $K(X) \otimes K(\mathbb{P}^n) \to K(X \times \mathbb{P}^n)$, we first need K-theoretic homotopy properties.

5.1 Homotopy Properties for $K(X)$

Proposition 5.4. *(Localization Sequence) Let X be a sub-variety (singular or non-singular) and let X' be a closed sub-variety. Set $U = X - X'$. We define a map $K(X') \to K(X)$ by the extension of a sheaf on X' by zero, and define a map $K(X) \to K(U)$ by sheaf restriction. Then the sequence*

$$K(X') \longrightarrow K(X) \longrightarrow K(U) \longrightarrow 0 \tag{5.14}$$

is exact.

Proposition 5.5. *If $Y = \mathbb{A}^1$, then the pullback homomorphism $p^* : K(X) \to K(X \times Y)$ is bijective.*

Proof. Consider $\mathscr{O}_X$ to be a sheaf on $X \times Y$. Then we have an exact sequence

$$0 \longrightarrow \mathscr{O}_{X\times Y} \xrightarrow{\iota} \mathscr{O}_{X\times Y} \longrightarrow \mathscr{O}_X \longrightarrow 0\ . \tag{5.15}$$

By the long exact sequence for Tor, $\mathrm{Tor}_p^{\mathscr{O}_{X\times Y}}(\mathscr{O}_X, \mathscr{F}) = 0$ for $p \geq 2$, provided that $\mathscr{F}$ is a coherent sheaf on $X \times Y$. Thus, we have a well-defined homomorphism

$$\pi : K(X \times Y) \to K(X), \tag{5.16}$$

given explicitly by

$$\pi(\mathscr{F}) = \mathrm{Tor}_0^{\mathscr{O}_{X\times Y}}(\mathscr{O}_X, \mathscr{F}) - \mathrm{Tor}_1^{\mathscr{O}_{X\times Y}}(\mathscr{O}_X, \mathscr{F}). \tag{5.17}$$

The composition $p^*\pi = \mathrm{id}$, so p^* is injective. We now consider $K(X) \subset K(X \times Y)$ via the pullback homomorphism.

Let $X' \subset X$ be a closed subvariety and let $U = X - X'$. We now demonstrate that $K(X) \simeq K(X \times Y)$ by induction on $n = \dim(X)$. The following diagram is commutative with exact rows (by the Localization Sequence):

$$\begin{array}{ccccccc} K(X') & \longrightarrow & K(X) & \longrightarrow & K(U) & \longrightarrow & 0\ . \\ \downarrow & & \downarrow & & \downarrow & & \\ K(X' \times Y) & \longrightarrow & K(X \times Y) & \longrightarrow & K(U \times Y) & \longrightarrow & 0 \end{array} \tag{5.18}$$

By hypotheses the left-most vertical arrow is an isomorphism. Thus, every element $z \in K(X \times Y)$ with $z_{U\times Y} \in K(U)$ is inside $K(X) \subset K(X \times Y)$. This allows us to ignore subvarieties of X with dimension less than n. In particular, we may suppose that $X = \mathrm{Spec}(A)$ is affine, non-singular, and irreducible. We then make use of the following lemma:

Lemma 5.6. *Let Z be an algebraic variety. The classes $[\mathscr{O}_T]$ generate $K(Z)$ when $T \subset Z$ is an irreducible subvariety.*

Proof. For a completely torsion coherent $\mathscr{O}_Z$-module $\mathscr{F}$, $\mathrm{supp}(\mathscr{F}) \subset T$. We need only consider $Z = \mathrm{Spec}(A)$, so then $\mathscr{F} = \widetilde{M}$, where M is an A-module. Then any torsion section $m \in M$ has support on $\mathrm{Spec}(A/(f))$, where $fm = 0$.

Now induct on $\dim(Z)$. Let $X \subset K(Z)$ be the subgroup generated by the structure sheaves of the irreducible subvarieties. Then, when $\mathscr{F}$ is a coherent $\mathscr{O}_X$-module, $[T(\mathscr{F})] \in X$, where $T(\mathscr{F})$ is the torsion subsheaf. Consider

$$0 \longrightarrow T(\mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow 0\ ; \tag{5.19}$$

we need to show that $[\mathscr{G}] \in X$. Let $k(Z)$ be the field of rational functions on Z. Then

$$\mathscr{F} \otimes k(Z) \cong k(Z)^n \tag{5.20}$$

for some n, so $\mathscr{G} \cong \mathscr{O}_Z^n$. Clearly, $[\mathscr{O}_Z^n] \in k(Z)$ so we are done. ☐

Applying this lemma, it suffices to show that $[\mathscr{O}_T] \in K(X)$ for all irreducible subvarieties $T \subset X \times Y$. By induction on $n = \dim(X)$, we consider the case where $\dim(T) = n$ and $\rho(T) \neq X$, where $\rho : X \times Y \to X$ is the projection. Let A be the coordinate ring of X and let $\mathfrak{p}$ be the prime ideal of the coordinate ing $A[t]$ of $X \times Y$ corresponding to T. Since $\rho(T)$ is dense in X, we have $A \cap \mathfrak{p} = 0$. Let $S \subset A$ be the set of invertible elements and $K = A_S$ be the field of fractions of A. Note that $A[t]_S = K[t]$. Since $\mathfrak{p} \cap S = \emptyset$, we can write $\mathfrak{p} = \mathfrak{p}' \cap A[t]$ for some non-zero prime $\mathfrak{p}'$ of $K[t]$. Thus, there is an irreducible polynomial $P(t)$ with coefficients in A such that $\mathfrak{p}$ is the set of polynomials in $A[t]$ that are divisible by $P(t)$ in $K[t]$. Then

$$\mathfrak{q} := A[t]P(t) \subset \mathfrak{p} \subset A[t]. \tag{5.21}$$

The sheaf $\mathscr{O}_t$ corresponds to the module $A[t]/\mathfrak{p}$. Let $\mathscr{F}$ be the sheaf corresponding to $A[t]/\mathfrak{q}$. By the equivalence $\mathfrak{p}_S = \mathfrak{q}_S$, there is an invertible $a \in A$ such that $a \cdot (\mathfrak{p} / \mathfrak{q}) = 0$, hence $\mathscr{O}_T$ is congruent, modulo an element of $K(X' \times Y)$ to $\mathscr{F}$ with $\dim(X') < \dim(X)$. This gives an exact sequence

$$0 \longrightarrow \mathscr{O}_{X\times Y} \xrightarrow{P(t)} \mathscr{O}_{X\times Y}\,\mathscr{F} \longrightarrow 0\ , \tag{5.22}$$

which shows that $[\mathscr{F}] = 0$ in $K(X \times Y)$. Then

$$[\mathscr{O}_T] \in \operatorname{im}(X' \times Y) = \operatorname{im} K(X') \subset K(X), \tag{5.23}$$

and we are done. □

Immediately, from induction on $n = \dim(X)$, we have:

Corollary 5.7. *If $Y = \mathbb{A}^1$, then $K(X) = K(X \times Y)$.*

5.2 GRR for $X \times \mathbb{P}^n \to X$

We now prove the surjectivity of $K(X) \otimes K(\mathbb{P}^n) \to K(X \times \mathbb{P}^n)$.

Proposition 5.8. *Let X be a variety. Then $K(X) \otimes K(\mathbb{P}^n) \to K(X \times \mathbb{P}^n)$ is a surjective homomorphism.*

Proof. We proceed by induction on n. The proposition is trivial for $n = 0$. For $n > 0$, let H be a hyperplane of $\mathbb{P}^n$ and let $U = \mathbb{P}^n - H$. Then the following diagram is commutative with exact rows:

$$\begin{array}{ccccccc} K(X)\otimes K(H) & \longrightarrow & K(X)\otimes K(\mathbb{P}^n) & \longrightarrow & K(X)\otimes K(U) & \longrightarrow & 0\ . \\ \downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi_2} & & \downarrow{\scriptstyle\varphi_3} & & \\ K(X\times H) & \longrightarrow & K(X\times \mathbb{P}^n) & \longrightarrow & K(X\times U) & \longrightarrow & 0 \end{array} \tag{5.24}$$

Note that φ_1 is surjective by the inductive hypothesis. φ_3 is as well, since Corollary (5.7) implies that $K(U) \simeq \mathbb{Z}$ and $K(X \times U) \simeq K(X)$. Thus, φ_2 is also surjective. □

To conclude the proof that $f : X \times \mathbb{P}^n \to X$ is projection onto the first factor, we verify a Hirzebruch-Riemann-Roch type result for $\mathbb{P}^n$:

Proposition 5.9. *Let $\mathscr{F}$ be a coherent sheaf on $\mathbb{P}^n$. Then*

$$\chi(\mathbb{P}^n, \mathscr{F}) = \int_{\mathbb{P}^n} \operatorname{ch}(\mathscr{F}) \operatorname{td}(T_{\mathbb{P}^n}). \tag{5.25}$$

Proof. Let $x = [H] \in A^1(\mathbb{P}^n)$ be the class of a hyperplane. Then

$$\operatorname{td}(T_{\mathbb{P}^n}) = \frac{x^{n+1}}{(1 - e^{-x})^{n+1}}. \tag{5.26}$$

By Theorem 4.8, we can write $[\mathscr{F}]$ as a $\mathbb{Z}$-linear combination of twisted structure sheaves. Thus, it suffices to prove our formula for the divisor sheaf $\mathscr{O}(r) = \mathscr{F}$ associated to rH. Computing the Chern character, we obtain:

$$\operatorname{ch}(\mathscr{O}(r)) = \operatorname{ch}(\mathscr{O}(1))^r = e^{rx}. \tag{5.27}$$

Remark 5.10. (Cohomology Computation) We recall that $H^\ell(\mathbb{P}^n, \mathscr{O}(r))$ is a vector space over a field k ([6] III.5) with the following properties:

1. $\dim H^\ell(\mathbb{P}^n, \mathscr{O}(r)) = 0$ for $0 < \ell < n$ and all $r \in \mathbb{Z}$.
2. $\dim H^0(\mathbb{P}^n, \mathscr{O}(r)) = \begin{cases} \binom{n+r}{r} & r \geq 0 \\ 0 & \textit{else.} \end{cases}$
3. $\dim H^n(\mathbb{P}^n, \mathscr{O}(r)) = \begin{cases} \binom{-r-1}{-n-r-1} & r \leq -n-1 \\ 0 & \textit{else.} \end{cases}$

Thus, the Euler characteristic is

$$\chi(\mathbb{P}^n, \mathscr{O}(r)) = \dim_k H^0(\mathbb{P}^n, \mathscr{O}(r)) + (-1)^n \dim_k H^n(\mathbb{P}^n, \mathscr{O}(r)) = \binom{n+r}{n}. \tag{5.28}$$

Now, we need only show that

$$\int_{\mathbb{P}^n} \mathrm{ch}(\mathscr{F})\, \mathrm{td}(T_{\mathbb{P}^n}) = \int \frac{e^{rx}x^{n+1}}{(1-e^{-x})^{n+1}} = \binom{n+r}{n}. \tag{5.29}$$

Expressing this formula in terms of residues, and letting $y = 1 - e^{-x}$, we obtain:

$$\int_{\mathbb{P}^n} \mathrm{ch}(\mathscr{O}(r))\, \mathrm{td}(T_{\mathbb{P}^n}) = \mathrm{res}_{x=0}\left(\frac{e^{rx}}{(1-e^{-x})^{n+1}}dx\right) = \mathrm{res}_{y=0}\left(\frac{(1-y)^{-r-1}}{y^{n+1}}dy\right) = \binom{n+r}{n}. \tag{5.30}$$

□

Corollary 5.11. *GRR is true for projection onto the first factor $X \times \mathbb{P}^n \to X$.*

5.3 GRR for Divisors

From Corollary 5.11, it suffices to show that GRR holds for closed immersions. Here we prove the intermediate case of the inclusion of a divisor. Then, we blow up along the divisor and prove that this is sufficient for the general case of a closed immersion.

Let Y be a closed sub-variety of X, let $i : Y \to X$ be the inclusion of Y in X, and let $\mathscr{J}$ be the ideal sheaf of Y. Then we have an exact sequence

$$0 \longrightarrow \mathscr{J} \longrightarrow \mathscr{O}_X \xrightarrow{r} \mathscr{O}_Y \longrightarrow 0 \tag{5.31}$$

where $r : \mathscr{O}_X \to \mathscr{O}_Y$ is the restriction map. We write

$$\mathscr{N} = \mathscr{N}_{Y,X} = \mathscr{H}om_{\mathscr{O}_Y}(\mathscr{J}/\mathscr{J}^2, \mathscr{O}_Y) \tag{5.32}$$

for the normal sheaf of Y in X. This is locally free since its dual $\mathscr{N}^* = \mathscr{J}/\mathscr{J}^2$ is. We write $p = \mathrm{codim}(Y, X)$.

Now, if D is a divisor on X, then $\mathscr{O}(D)$ is the invertible sheaf determined by D. We construct this locally on $\mathrm{Spec}(A) \subset X$ as the rank one subsheaf of the field of fractions sheaf of A. This is generated by the inverse of a local defining function for D. The relation

$$c(\mathscr{O}(D)) = 1 + [D] \tag{5.33}$$

holds.

5.3.1 Algebraic Interlude: Koszul Complexes

Definition 5.12. Let A be a Noetherian ring. For any A-module M, $x \in A$ is M-regular if $x : M \to M$ is injective. A sequence $x_1, \dots, x_n \in A$ is a *regular sequence* for M if:

1. x_1 is regular as an element, and x_i is $(M/(x_1, \dots, x_{i-1})M)$-regular for all i.
2. $M/(x_1, \dots, x_n)M \neq 0$.

Example 5.13. If $F \neq 0$, $G \neq 0$ are homogeneous polynomials of positive degree with $\gcd(F, G) = 1$ in $S = k[X_0, \dots, X_n]$ then F, G are S-regular.

We use regular sequences to define the *Koszul Complex* of an A-module M and a sequence $x_1, \dots, x_r \in A$. Let E be a free A-module with basis $e_1, \dots, e_r$. Then the Koszul complex (for M) is the sequence

$$0 \longrightarrow M \otimes \bigwedge^r E \xrightarrow{d} M \otimes \bigwedge^{r-1} E \longrightarrow \cdots \longrightarrow M \otimes E \longrightarrow M \tag{5.34}$$

with A-linear differential

$$d(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_j (-1)^{j-1} x_{i_j} e_{i_1} \wedge \cdots \wedge e_{i_p}. \tag{5.35}$$

When $M = A$ the entire Koszul complex is exact. In general we have the following

Theorem 5.14. *If* $\underline{x} = x_1, \dots, x_n$ *is an* M*-regular sequence, then*

$$H_p(K.(\underline{x}, M)) = \begin{cases} 0, & p \neq 0 \\ M/(\underline{x}M), & p = 0 \end{cases} \tag{5.36}$$

for Koszul complex $K.$*.*

This construction is of consequence for our theorem because we can replace A by $\mathscr{O}_X$ and M by an $\mathscr{O}_X$-module $\mathscr{F}$. We then obtain a locally free resolution of $\mathscr{F}$. In the following Proposition, we prove the case $\mathscr{F} = \mathscr{O}_Y$.

Proposition 5.15. *Let* $Y_1, \dots, Y_m$ *be non-singular sub varieties of* X *such that*

$$Y_{i-1} \cap \cdots \cap Y_1 \tag{5.37}$$

meets Y_i *transversely for* $i = 2, \dots, m$*. Then in* $K(X)$

$$[\mathscr{O}_{Y_1 \cap \cdots \cap Y_m}] = \prod_i [\mathscr{O}_{Y_i}]. \tag{5.38}$$

Remark 5.16. We say that two subvarieties Y_i, Y_j of X meet *transversely* if the set of defining functions for both generate the maximal ideal of the stalk $\mathscr{O}_{X,x}$ at any point $x \in Y_i \cap Y_j$.

Proof. By induction, it suffices to prove the case where two sub varieties Y, Z meet transversely. Definitionally, $[\mathscr{O}_Y] \cdot [\mathscr{O}_Z]$ is the alternating sum of $\mathrm{Tor}_i^{\mathscr{O}_X}(\mathscr{O}_Y, \mathscr{O}_Z)$. If $Y \cap Z = 0$, then both the left and right hand sides of (5.38) are clearly zero.

Now take some $a \in Y \cap Z$. Write local defining functions $f_1, \dots, f_p$ for Y and $g_1, \dots, g_q$ for Z about a. We tensor the Koszul complex for $\mathscr{O}_Y$ with $\mathscr{O}_Z$ to obtain the exact sequence

$$0 \longrightarrow \mathscr{O}_Z \otimes_k \bigwedge^p E \longrightarrow \cdots \longrightarrow \mathscr{O}_Z \otimes_k E \longrightarrow \mathscr{O}_Z\ . \tag{5.39}$$

As elements of $\mathscr{O}_{Z,a} = \mathscr{O}_{X,a}/(g_1, \dots, g_q)$, the f_i form a regular sequence. Thus, the Koszul complex homology gives the derived functors $\mathrm{Tor}_i^{\mathscr{O}_X}(\mathscr{O}_Y, \mathscr{O}_Z)$. By Theorem 5.14, all homology groups vanish except for $p = 0$, which is precisely $\mathscr{O}_{Z,a}/(f_1, \dots, f_p) = \mathscr{O}_{Y \cap Z, a}$. Hence

$$\mathrm{Tor}_0 = [\mathscr{O}_Y] \cdot [\mathscr{O}_Z] = [\mathscr{O}_{Y \cap Z}] \tag{5.40}$$

as desired. □

Corollary 5.17. *Let Y be a non-singular hyperplane section of X and let k be the dimension of X. Then $(1-[\mathscr{O}_Y])^{k+1}=0$.*

Proof. We know that $[\mathscr{O}_{Y_1}]=[\mathscr{O}_{Y_1}]$ for any two non-singular hyperplane sections of X. Furthermore, by the exact sequence

$$0 \longrightarrow \mathscr{O}(Y)^{-1} \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_Y \longrightarrow 0 \tag{5.41}$$

we have $[\mathscr{O}_Y]=1-[\mathscr{O}(Y)]^{-1}$. Applying the previous proposition, $(1-[\mathscr{O}(Y)]^{-1})^{k+1}=0$. Multiplying through by $\pm[\mathscr{O}(Y)]^{k+1}$ gives the desired result. □

Proposition 5.18. *For any $y \in K(Y)$,*

$$i^* i_*(y) = y \cdot \lambda_{-1}(\mathscr{N}^*). \tag{5.42}$$

In particular, $i^*[\mathscr{O}_Y]=\lambda_{-1}(\mathscr{N}^*)$.

Proof. By linearity it suffices to prove for the case where $y=[\mathscr{F}]$ for some locally free sheaf $\mathscr{F}$. Since $\mathscr{F}$ is locally free $i_*[\mathscr{F}]=[i_*\mathscr{F}]$, so we write

$$i^* i_*[\mathscr{F}] = \sum_{j\geq 0}(-1)^j[\mathrm{Tor}_j^{\mathscr{O}_X}(\mathscr{F},\mathscr{O}_Y)]. \tag{5.43}$$

Also, by the freeness of $\mathscr{F}$ and $\lambda^i(\mathscr{N}^*)$, we have

$$[\mathscr{F}]\cdot\lambda_{-1}(\mathscr{N}^*) = [\mathscr{F}]\cdot\left(1-[\mathscr{N}^*]t+[\wedge^2\mathscr{N}^*]t^2-\dots\right) \tag{5.44}$$

$$= [\mathscr{F}]\cdot[\mathscr{F}\otimes_{\mathscr{O}_Y}\mathscr{N}^*]t+[\mathscr{F}\otimes_{\mathscr{O}_Y}\wedge^2\mathscr{N}^*]t^2-\dots \tag{5.45}$$

From the normal sheaf isomorphism $\mathscr{N}\simeq\mathscr{J}/\mathscr{J}^2$, it suffices to prove

1. $\mathrm{Tor}_1^{\mathscr{O}_X}(\mathscr{F},\mathscr{O}_Y)=\mathscr{F}\otimes_{\mathscr{O}_Y}\mathscr{J}/\mathscr{J}^2$
2. $\mathrm{Tor}_i^{\mathscr{O}_X}(\mathscr{F},\mathscr{O}_Y)=\mathscr{F}\otimes_{\mathscr{O}_Y}\wedge^i\mathrm{Tor}_1^{\mathscr{O}_X}(\mathscr{O}_Y,\mathscr{O}_Y)=\mathscr{F}\otimes_{\mathscr{O}_Y}\lambda^i\left(\mathrm{Tor}_1^{\mathscr{O}_X}(\mathscr{O}_Y,\mathscr{O}_Y)\right)$.

The exterior product in (2) is well-defined by virtue of the local freeness of $\mathrm{Tor}_1^{\mathscr{O}_X}(\mathscr{O}_Y,\mathscr{O}_Y)$; thus, (1) must be proved first.

We apply the long exact sequence for Tor to (5.31) and obtain the exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^{\mathscr{O}_X}(\mathscr{F},\mathscr{O}_Y) \longrightarrow \mathscr{J}\otimes_{\mathscr{O}_X}\mathscr{F} \xrightarrow{\varphi} \mathscr{F}, \tag{5.46}$$

where φ is given by $\varphi(u\otimes v)=uv$. We know that $\mathscr{F}$ is supported on Y and is annihilated by $\mathscr{J}$. Thus, the middle arrow is an isomorphism. Furthermore, the image of $\mathscr{J}^2\otimes\mathscr{F}$ is zero in $\mathscr{J}\otimes\mathscr{F}$, so we have an isomorphism with $\mathscr{F}\otimes_{\mathscr{O}_X}\mathscr{J}/\mathscr{J}^2$. Finally, the tensor product is unchanged over $\mathscr{O}_Y$ rather than $\mathscr{O}_X$, so we have (1).

To prove (2), we use the Koszul complex from the previous proposition. We know the homology groups for the Koszul complex for the sheaf $\mathscr{F}$ are precisely $\mathrm{Tor}_i^{\mathscr{O}_X}(\mathscr{F},\mathscr{O}_Y)$ by the definition of Tor as a derived functor. Now, the local defining functions for Y are the sections of $\mathscr{J}$ and annihilate $\mathscr{F}$, so the differentials in the complex are all zero. This gives

$$\mathrm{Tor}_i^{\mathscr{O}_X}(\mathscr{F},\mathscr{O}_Y)=\mathscr{F}\otimes_k\wedge^i E=\mathscr{F}\otimes_{\mathscr{O}_Y}(\mathscr{O}_Y\otimes_k\wedge^i E). \tag{5.47}$$

That is, the homology groups and the terms of the complex are the same. We take $\mathscr{F}=\mathscr{O}_Y$ and $i=1$, so we have $\mathrm{Tor}_1^{\mathscr{O}_X}(\mathscr{O}_Y,\mathscr{O}_Y)=\mathscr{O}_Y\otimes_k E$. From the above equality, we obtain

$$\mathrm{Tor}_i^{\mathscr{O}_X}(\mathscr{F},\mathscr{O}_Y)=\mathscr{F}\otimes_{\mathscr{O}_Y}\bigwedge^i\mathrm{Tor}_1^{\mathscr{O}_X}(\mathscr{O}_Y,\mathscr{O}_Y). \tag{5.48}$$

This completes the proof. □

Proposition 5.19. *Let D be a divisor on X and $\mathscr{L} = \mathscr{O}(D)\mid_D$. Then*

1. $\mathscr{L} = \mathscr{N}_{D,X}$.
2. $[\mathscr{O}_D] = 1 - [\mathscr{O}(D)]^{-1}$.
3. $i^* i_* = y \cdot (1 - [\mathscr{L}]^{-1})$ *for all* $y \in K(D)$.

Proof. 1. Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open cover of X such that the divisor D is given by $f_i = 0$ on U_i and $df_i \neq 0$ on $D \cap U_i$. For $U_{ij} = U_i \cap U_j$ take $g_{ij} = f_i/f_j$ as the transition function. Note that $\mathscr{N}^*$ is trivial on U_i, but $df_i = g_{ij} df_j$ on $D \cap U_{ij}$, so $\mathscr{N}^*$ is defined on $\mathfrak{U}$ by g_{ij}^{-1}.

2. Consider the exact sequence

$$0 \longrightarrow \mathscr{J} \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_D \longrightarrow 0 \tag{5.49}$$

and take $\mathscr{J} = \mathscr{L}(D)^*$. The result follows immediately.

3. This is simply an application of the previous proposition and (1).

□

It remains to be shown that

$$\mathrm{ch}(i_*(y)) = i_*(\mathrm{ch}(y) \cdot \mathrm{td}(\mathscr{N})^{-1}), \tag{5.50}$$

which is sufficient to demonstrate our case of the closed immersion. Indeed, from the following exact sequence

$$0 \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_{X|Y} \longrightarrow \mathscr{N}_{Y,X} \longrightarrow 0 \tag{5.51}$$

we have that $i^* \mathrm{td}(T_X) = \mathrm{td}(T_Y) \cdot \mathrm{td}(\mathscr{N})$. Thus, for any $y \in K(Y)$,

$$i_*(\mathrm{ch}(y) \cdot \mathrm{td}(\mathscr{N})^{-1}) = i_*(\mathrm{ch}(y) \cdot \mathrm{td}(T_Y) \cdot i^*(\mathrm{td}(T_X)^{-1})) = i_*(\mathrm{ch}(y) \cdot \mathrm{td}(T_Y)) \cdot \mathrm{td}(T_X)^{-1}, \tag{5.52}$$

and (5.50) suffices. We prove this for the special case of $Y = D$ and divisor on X.

Proposition 5.20. *Equation (5.50) holds for $Y = D$ a divisor on X and $y = i^*(x)$ where $x \in K(X)$.*

Proof. We use the projection formula:

$$f_*(y \cdot f^*(x)) = f_*(y) \cdot x \tag{5.53}$$

and (2) in Proposition 5.19 to obtain

$$\mathrm{ch}(i_* i^*(x)) = \mathrm{ch}(x \cdot i_*(1)) = \mathrm{ch}(x \cdot (1 - [D]^{-1})). \tag{5.54}$$

Recall that $x \to \mathrm{ch}(x)$ is a ring homomorphism and that (as noted above) $c(\mathscr{O}(D)) = 1 + [D]$. Set $y = i^*(x)$, so the left-hand side of (5.50) becomes

$$\mathrm{ch}(i_*(y)) = \mathrm{ch}(x)\,\mathrm{ch}(1 - [\mathscr{O}(D)]^{-1}) = \mathrm{ch}(x) \cdot (1 - e^{-[D]}). \tag{5.55}$$

Furthermore, the right-hand side of (5.50) becomes

$$i_*(\mathrm{ch}(i^*(x)) \cdot \mathrm{td}(\mathscr{O})^{-1}) = i_*(i^* \mathrm{ch}(x) \cdot i^* \mathrm{td}(\mathscr{O}(D))^{-1}) \tag{5.56}$$

$$= \mathrm{ch}(x) \cdot \mathrm{td}(\mathscr{O}(D))^{-1} \cdot i_*(1) \tag{5.57}$$

$$= \mathrm{ch}(x) \cdot \mathrm{td}(\mathscr{O}(D))^{-1} \cdot [D]. \tag{5.58}$$

Finally, we know that $\mathrm{td}(\mathscr{O}(D)) = [D] \cdot (1 - e^{-[D]})^{-1}$, so we have (5.51). □

Remark 5.21. This final term $[D] \cdot (1 - e^{-[D]})^{-1}$ is precisely the failure of commutativity of $\mathrm{ch}(\cdot)$ and f_*. This accounts for the appearance of the Todd class in more general versions of GRR.

5.4 Blowing Up Along the Divisor

Corollary 5.22. *GRR is true for closed immersion $i : Y \to X$ where $X = Y \times \mathbb{P}^n$ and i maps $a \in Y \mapsto (a, p)$, where $p \in \mathbb{P}^n$.*

Proof. By Lemma 5.3 and the fact that GRR is true for the identity $i : Y \to Y$, we need only prove the proposition for $i : \{a\} \to \mathbb{P}^n$. Finally, since $K(\{a\}) \simeq \mathbb{Z}$, we consider $1 \in K(\{a\})$.

By Proposition 5.20 we know that GRR holds for the divisor case ($n = 1$). Thus, by induction, we consider the map $u : \{a\} \to H$ for some hyperplane H of projective space. It suffices to prove that GRR is true for $v : H \to \mathbb{P}^n$. This will follow from the divisor case once we have that $u_*(1) \in v^*(K(\mathbb{P}^n))$.

Let Z be another hyperplane in $\mathbb{P}^n$ and L a line in H such that $\{a\} = L \cap Z \cap H$. By Proposition 5.15, we have that

$$[\mathscr{O}_Y] = [\mathscr{O}_L] \cdot [\mathscr{O}_{H \cap Z}]. \tag{5.59}$$

Then, by (2) of Proposition 5.19, we have $[\mathscr{O}_{H\cap Z}] = 1 - [\mathscr{O}(H \cap Z)]^{-1}$. Furthermore, as Z is a hyperplane, we have the identification $\mathscr{O}(H \cap Z) = \mathscr{O}(H) \mid_H$. Finally, (3) of Proposition 5.19 gives

$$u_*(1) = [\mathscr{O}_Y] = v^* v_* [\mathscr{O}_L] \tag{5.60}$$

with $[\mathscr{O}_Y] \in K(H)$ and $v_*[\mathscr{O}_L] \in K(\mathbb{P}^n)$. □

Corollary 5.23. *If Equation (5.50) holds for $2\dim(Y) \leq \dim(X) - 2$, then it holds in general.*

Proof. We have that GRR is true for $X \to X \times \mathbb{P}^n$. Form the composition $Y \to X \to X \times \mathbb{P}^n$ and use (2) of Lemma 5.2 to see that GRR is true for $Y \to X$. Simply select n sufficiently large enough such that $\dim(X) + n - 2 \geq 2\dim(Y)$ to finish the proof. □

We consider the blow-up commutative diagram:

$$\begin{array}{ccc} Y' & \xrightarrow{j} & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{i} & X \end{array} \tag{5.61}$$

where X is a non-singular variety, $i : Y \to X$ is the inclusion of the nonsingular subvariety Y into X. Furthermore, $f : X' \to X$ is the blowup of X along Y, $j : Y' \to X'$ is the inclusion of the exceptional divisor, and g is the restriction $f\mid_{Y'}$. As before, $p = \operatorname{codim}(Y, X)$, so g becomes a projective bundle with fiber $\mathbb{P}^{p-1}$. We denote the normal bundle of Y in X by N and $\widetilde{N} = g^*N$ its pullback to Y'. Let L be the line bundle corresponding to the exceptional divisor Y' in X'. Finally, define $F = \widetilde{N}/L$, which is a locally free sheaf of rank $p - 1$.

We will prove that GRR is true for a closed immersion $i : Y \to X$ of two non-singular, quasi-projective varieties using a series of lemmata.

Lemma 5.24. *Let G be a rank k vector bundle over a variety X. Then*

$$\operatorname{ch}(\lambda_{-1}G) = c_k(G^*)\operatorname{td}(G^*)^{-1} \tag{5.62}$$

Proof. We write

$$c(G) = \prod_{i=1}^{k}(1 + a_i) \tag{5.63}$$

by the Splitting Principle. Furthermore,

$$c(G^*) = \prod_{i=1}^{k}(1-a_i), \qquad \text{and} \qquad c(\wedge^j G) = \prod_{i_1<\cdots<i_j}(1+a_{i_1}+\cdots+a_{i_j}). \tag{5.64}$$

Thus, the top Chern class for G^* is

$$c_k(G^*) = (-1)^k a_1, \ldots, a_k. \tag{5.65}$$

Finally, from the definition of the Todd class

$$\mathrm{ch}(\lambda_{-1}(G)) = \prod_{i=1}^{k}(1-e^{a_i}) = \mathrm{td}(G^*)c_k(G^*). \tag{5.66}$$

□

The next four lemmata are particular to the blow-up diagram.

Lemma 5.25. *We have that* $f_*(1) = 1$ *for* $f_* : A*(X') \to A^*(X)$. *Then* $f_*f^* = id_{A^*(X)}$.

Proof. The map $f : X' \to X$ is an isomorphism except at the exceptional divisor. As such, it has a local degree 1. This tells us that f_* maps $1 \in A^*(X') \mapsto 1 \in A^*(X)$, i.e., fundamental cycles to fundamental cycles. □

Lemma 5.26. *The proper pushforward* $g_* : A^*(Y') \to A(Y)$ *satisfies* $g_*(c_{p-1}(F)) = 1$.

Proof. The pushforward g_* lowers the geometric codimension, i.e., degree by $\dim(Y')-\dim(Y) = p-1$, which corresponds to integration on the fiber. Then the restriction of L to the fiber $\mathbb{P}^{p-1}$ is isomorphic to $k^p - \{0\}$, i.e., the principal fiber of the group k^* with base $\mathbb{P}^{p-1}$. Then, the first Chern class is $c_1(L) = -[H]$ where H is a hyperplane in the fiber $\mathbb{P}^{p-1}$. We have that $g_*([H]^{p-1}) = 1$ and $g_*([H])^i = 0$ for $0 \le i < p-1$ by dimensional computation. Recall that $\widetilde{N}/L = F$, so $c(\widetilde{N}) = (1-[H])c(F)$ and $c(F) = g^*c(N)\cdot(1+[H]+[H]^2+\cdots)$. Then

$$c(F) = c(\widetilde{N})\cdot(1-[H])^{-1} = g^*c(N)\cdot(1+[H]+[H]^2+\cdots). \tag{5.67}$$

Hence,

$$c_{p-1}(F) = [H]^{p-1} + g^*(c_1(N))\cdot[H]^{p-2} + \cdots + g^*(c_{p-1}(N)). \tag{5.68}$$

We apply the pushforward to obtain

$$g_*(c_{p-1}(F)) = g_*([H]^{p-1}) + c_1(N)g_*([H]^{p-2}) + \cdots + c_{p-1}(N)g_*(1). \tag{5.69}$$

Thus, $g_*(c_{p-1}(F)) = 1$. □

Lemma 5.27. *For all* $y \in K(Y)$, *we have* $f^*i_*(y) = j_*(g^*(y)\lambda_{-1}(F^*))$.

Proof. We continue to write $\mathscr{J} = \mathscr{J}_Y$ for the ideal sheaf of $Y \subset X$ and $\mathscr{J}' = \mathscr{J}_{Y'}$ for the ideal sheaf of $Y' \subset X'$. Furthermore, $\mathscr{J}/\mathscr{J}^2$ is the sheaf of germs of sections of E^* and $\mathscr{J}'/\mathscr{J}^2$ is the sheaf of germs of sections of L^*. Finally, $\mathscr{J}/\mathscr{J}^2 \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y'}$ is the sheaf of germs of sections of $g^*(E^*) = E'^*$. Taking an element $u \in \mathscr{J}_x$ mapping to $u \circ f \in \mathscr{J}_{f^{-1}(x)}$ defines a surjective homomorphism of $\mathscr{O}_{Y'}$-modules

$$\mu : \mathscr{J}/\mathscr{J}^2 \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y'} \longrightarrow \mathscr{J}'/\mathscr{J}'^2 \tag{5.70}$$

This corresponds to a map $\mu' : E'^* \to L^*$ and thus an injection $L \hookrightarrow E'$. We have that $\ker(\mu)$ is the sheaf $\mathscr{O}_{Y'}(\mathscr{F}^*)$ of germs of sections of F^*, which is locally free. Then $E'^*/\ker(\mu) = L^*$ and $[\ker(\mu)] = [F^*]$ in $K(Y')$. Therefore, we have an exact sequence

$$0 \longrightarrow \mathscr{O}_{Y'}(F^*) \longrightarrow \mathscr{J}/\mathscr{J}^2 \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y'} \xrightarrow{\mu} \mathscr{J}'/\mathscr{J}'^2 \longrightarrow 0. \tag{5.71}$$

By linearity, we need only prove the lemma for $y = [\mathscr{G}]$, where $\mathscr{G}$ is a locally free sheaf. Then $g^*(y) = [\mathscr{G} \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y'}]$ is locally free on Y' and

$$g^*(y) \cdot \lambda_{-1}(F^*) = \sum_{i \geq 0} (-1)^i [\mathscr{G} \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y'} \otimes_{\mathscr{O}_{Y'}} \mathscr{O}_{Y'}(\lambda^i[F^*])] \tag{5.72}$$

$$= \sum_{i \geq 0} (-1)^i [\mathscr{G} \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y'}(\lambda^i[F^*])]. \tag{5.73}$$

Also, $f^* i_*(y)$ is the alternating sum of $\operatorname{Tor}_i^{\mathscr{O}_X}(\mathscr{G}, \mathscr{O}_{X'})$. Thus, our proof of the lemma reduces to the following Tor equalities:

1. $\operatorname{Tor}_i^{\mathscr{O}_X}(\mathscr{O}_Y, \mathscr{O}_{X'}) = \lambda^i \operatorname{Tor}_1^{\mathscr{O}_X}(\mathscr{O}_Y, \mathscr{O}_{X'}) \qquad (i \geq 1)$.
2. $[\operatorname{Tor}_i^{\mathscr{O}_X}(\mathscr{O}_Y, \mathscr{O}_{X'}) = [\mathscr{O}_{Y'}(F^*)]]$.
3. $\operatorname{Tor}_j^{\mathscr{O}_X}(\mathscr{G}, \mathscr{O}_{X'}) = \mathscr{G} \otimes_{\mathscr{O}_Y} \operatorname{Tor}_j^{\mathscr{O}_X}(\mathscr{O}_Y, \mathscr{O}_{X'}) \qquad (j \geq 1)$.

We prove (1) using the Koszul complex

$$0 \longrightarrow \mathscr{O}_{X'} \otimes_k \bigwedge\nolimits^p E \longrightarrow \cdots \longrightarrow \mathscr{O}_{X'} . \tag{5.74}$$

We need only consider a neighborhood of $b' \in Y'$, as both sides of (1) vanish outside of Y'. We take $U \subset X$ open and containing $b = g(b')$. The local coordinate expression of the blow-up is given by

$$U' = f^{-1}(U) = \{(x, y) : x_i f_j(y) - x_j f_i(y) = 0\}, \tag{5.75}$$

where $f_1, \ldots, f_p$ locally define $Y \cap U$ in U and $[x_0 : \ldots : x_{p-1}]$ are the homogeneous coordinates for fibers $\mathbb{P}^{p-1}$ of f. We let $\mathfrak{U} = \{U_i\}$ be the affine open cover of $\mathbb{P}^{p-1}$, so then $U'_i = f^{-1}(U_i \cap U)$ form an open cover of U'. We take $b' \in U'_j$ and observe that our Koszul complex becomes

$$0 \longrightarrow \mathscr{O}_{X'} \otimes_k \bigwedge\nolimits^p E' \longrightarrow \cdots \longrightarrow \mathscr{O}_{X'} , \tag{5.76}$$

where E' has basis $\{e'_i\}$ and differential

$$d(1 \otimes e'_j) = f_j \otimes 1 \qquad d(1 \otimes e'_i) = \left(f_i - f_j \frac{x_i}{x_j} \right) \otimes 1, \tag{5.77}$$

when $i \neq j$. Then the cycles of this complex (of exterior power s) are precisely

$$Z_s = \mathscr{O}_{X'} \otimes_k \bigwedge^s (e'_1, \ldots, \widehat{e}'_j, \ldots, e'_p) \tag{5.78}$$

and the boundaries are

$$B_s = f_j \cdot \mathscr{O}_{X'} \otimes_k \bigwedge^s (e'_1, \ldots, \widehat{e}'_j, \ldots, e'_p) \,. \tag{5.79}$$

Thus,

$$\mathrm{Tor}_i^{\mathscr{O}_X}(\mathscr{O}_Y, \mathscr{O}_{X'}) \simeq \mathscr{O}_{Y'} \otimes_k \bigwedge^i (e'_1, \ldots, \widehat{e'_j}, \ldots, e'_p)\,. \tag{5.80}$$

This proves (1).

We now prove (2). From the short exact sequence

$$0 \longrightarrow \mathscr{J} \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_Y \longrightarrow 0\,, \tag{5.81}$$

we get

$$0 \longrightarrow \mathrm{Tor}_1^{\mathscr{O}_X}(\mathscr{O}_Y, \mathscr{O}_{X'}) \longrightarrow \mathscr{J} \otimes_{\mathscr{O}_X} \mathscr{O}_{X'} \xrightarrow{\ g\ } \mathscr{O}_{X'} \tag{5.82}$$

in the associated long exact sequence since $\mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{O}_{X'} = \mathscr{O}_{X'}$ and $\mathrm{Tor}_1^{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{O}_{X'})$ vanishes. We want to show that g in the above sequence is precisely our surjective homomorphism μ. Note that

$$\mathrm{Tor}_1^{\mathscr{O}_X}(\mathscr{O}_Y, \mathscr{O}_{X'}) \otimes_{\mathscr{O}_{X'}} \mathscr{O}_{Y'} = \mathrm{Tor}_1^{\mathscr{O}_X}(\mathscr{O}_Y, \mathscr{O}_{X'}) \otimes_{\mathscr{O}_{Y'}} \mathscr{O}_{Y'} = \mathrm{Tor}_1^{\mathscr{O}_X}(\mathscr{O}_Y, \mathscr{O}_{X'}). \tag{5.83}$$

Then the defining short exact sequence for $\mathscr{J}'$, analogous to (5.82), has

$$\mathrm{Tor}_1^{\mathscr{O}_{X'}}(\mathscr{J}', \mathscr{O}_{Y'}) \longrightarrow \mathrm{Tor}_1^{\mathscr{O}_X}(\mathscr{O}_Y, \mathscr{O}_{X'}) \otimes_{\mathscr{O}_{X'}} \mathscr{O}_{Y'} \longrightarrow \mathscr{J} \otimes_{\mathscr{O}_X} \mathscr{O}_{X'} \otimes_{\mathscr{O}_{X'}} \mathscr{O}_{Y'} \longrightarrow \mathscr{J}' \otimes_{\mathscr{O}_{X'}} \mathscr{O}_{Y'} \longrightarrow 0 \tag{5.84}$$

in its associated long exact sequence for $\mathrm{Tor}^{\mathscr{O}_{X'}}(\cdot, \mathscr{O}_{X'})$. Also, $\mathscr{J}' = \mathscr{O}_{X'}([Y']^{-1})$, which is locally free, and $\mathrm{Tor}_1^{\mathscr{O}_{X'}}(\mathscr{J}', \mathscr{O}_{Y'}) = 0$. This, along with the above piece of the Tor long exact sequence and (5.84) gives the short exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^{\mathscr{O}_X}(\mathscr{O}_Y, \mathscr{O}_{X'}) \longrightarrow \mathscr{J} \otimes_{\mathscr{O}_X} \mathscr{O}_{Y'} \xrightarrow{\ g\ } \mathscr{J}' \otimes_{\mathscr{O}_{X'}} \mathscr{O}_{Y'} \longrightarrow 0. \tag{5.85}$$

Hence the image of $\mathscr{J}^2 \otimes_{\mathscr{O}_X} \mathscr{O}_{Y'}$ in $\mathscr{J} \otimes_{\mathscr{O}_X} \mathscr{O}_{Y'}$ is zero as is the image of $\mathscr{J}'^2 \otimes_{\mathscr{O}_{X'}} \mathscr{O}_{Y'}$ in $\mathscr{J}' \otimes_{\mathscr{O}_{X'}} \mathscr{O}_{Y'}$. Then we have

$$\mathscr{J} \otimes_{\mathscr{O}_X} \mathscr{O}_{Y'} = \mathscr{J} \,/\, \mathscr{J}^2 \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y'}, \qquad \mathscr{J}' \otimes_{\mathscr{O}_{X'}} \mathscr{O}_{Y'} = \mathscr{J}' \,/\, \mathscr{J}'^2, \tag{5.86}$$

thus verifying the equivalence of μ and g. Finally, comparing this to (5.72) gives (2) as an equality of sheaves.

Finally, we prove (3). Let

$$T(\mathscr{G}, \mathscr{O}_{X'}) = \mathscr{G} \otimes_{\mathscr{O}_Y} \mathscr{O}_Y \otimes \mathscr{O}_{X'}, \tag{5.87}$$

which is functorial in $\mathscr{G}$ and $\mathscr{O}_{X'}$. We wish to compute the left derived functors L_iT either in $\mathscr{G}$ or $\mathscr{O}_{X'}$. We use the E_2 spectral terms

$$E_2^{ij} = \mathrm{Tor}_i^{\mathscr{O}_Y}(\mathrm{Tor}_j^{\mathscr{O}_X}(\mathscr{O}_{X'}, \mathscr{O}_Y), \mathscr{G}), \qquad E_2'^{ji} = \mathrm{Tor}_j^{\mathscr{O}_X}(\mathrm{Tor}_i^{\mathscr{O}_Y}(\mathscr{G}, \mathscr{O}_Y), \mathscr{O}_{X'}). \tag{5.88}$$

Indeed, for some $i > 0$, these terms will coincide. Thus,

$$E_2^{0j} = E_2'^{j0} = L_jT(\mathscr{G}, \mathscr{O}_{X'}). \tag{5.89}$$

Finally, $E_2^{0j} = \mathscr{G} \otimes_{\mathscr{O}_Y} \mathrm{Tor}_j^{\mathscr{O}_X}(\mathscr{O}_Y, \mathscr{O}_{X'})$ and $E_2'^{j0} = \mathrm{Tor}_j^{\mathscr{O}_X}(\mathscr{G}, \mathscr{O}_{X'})$, so (3) holds.

□

We require one more lemma in order to finish our proof of GRR:

Lemma 5.28. *If $p \geq \dim(Y)+2$, then $\lambda_{-1}(F^*) \equiv 0 \mod (1-L^*)$.*

However, the proof of this lemma requires a further result on vector bundles:

Lemma 5.29. *Let $q = \dim(Y)$ and let G be a vector bundle of rank $p = q+k$ on Y, where $k \geq 0$. Then $\lambda^s([G]-k) = 0$ for $s \geq q+1$.*

Proof. Let ℓ be the line bundle of a hyperplane section of Y. Then, by Proposition 5.15, we have that

$$(1-[\ell])^{q+1} = 0, \tag{5.90}$$

hence $[\ell] = 1+u \in K(Y)$ satisfies $u^{q+1} = 0$, and

$$[\ell]^n = \sum_{i=0}^{q} \binom{n}{i} u^i. \tag{5.91}$$

We compute the exterior powers and see that

$$\lambda_t([G]\cdot[\ell]^n - k) = \prod_{i=1}^{q} \lambda_t([G]\cdot u^i)^{\binom{n}{i}} \cdot (1-t)^{-k}. \tag{5.92}$$

Then we see that the respective coefficients are of the form

$$\lambda^s([G]\cdot[\ell]^n - k) = \sum_{i=1}^{m_s} B_{s,i} P_{s,i}(n), \tag{5.93}$$

where $B_{s,i} \in K(Y)$ and the $P_{s,i}(n)$ are polynomials with rational coefficients and integer values when n is sufficiently large. A theorem of Hilbert states that the $P_{s,i}$ can be expressed as a $\mathbb{Z}$-linear combination of

$$\binom{x}{j} = x(x-1)\cdots\frac{(x-j+1)}{j!}, \tag{5.94}$$

so we write

$$\lambda^s([G]\cdot[\ell]^n - k) = \sum_{i=0}^{n_s} A_{s,i}\binom{n}{i}, \tag{5.95}$$

where $A_{s,i} \in K(Y)$. For $n > n_0$, the bundle $G \otimes \ell^n$ is ample, so it contains a trivial fiber of rank k. Hence, we have the identification $[G]\cdot[\ell]^n - k = [G']$ for some fiber G' of rank q. Also, $\lambda^s[G'] = 0$ for $s \geq q+1$.

Now it suffices to prove the following: if

$$P(n) = \sum_{i=0}^{m} A_i \binom{n}{i} = 0 \tag{5.96}$$

with $A_i \in K(Y)$ for $n > n_0$, then $A_i = 0$ for all i. We induct on m and consider the difference between consecutive values of P

$$P(n+1) - P(n) = \sum_{i=0}^{m} A_i \left(\binom{n+1}{i} - \binom{n}{i}\right) = \sum_{i=0}^{m} A_i \binom{n}{i-1} = \sum_{0}^{m-1} A_{j+1}\binom{n}{j}. \tag{5.97}$$

Since $P(n+1)-P(n) = 0$ for $n > n_0$, the A_i vanish by the inductive hypothesis and the observation that $A_0 = 0$. □

Proof. (Lemma 5.28) First we prove two necessary formulae:

1. $\lambda^k([\mathscr{G}]-1) = (-1)^k \lambda_{-1}[\mathscr{G}]$.
2. $\lambda_t([\mathscr{G}]\cdot(1-[\mathscr{L}])) \equiv 1 \mod (1-[\mathscr{L}])$.

Let $\mathscr{G}$ be a locally free sheaf of rank k on Y. Then

$$\lambda_t([\mathscr{G}]-1) = \lambda_t[\mathscr{G}]/\lambda_t(1) = \lambda_t[\mathscr{G}]\cdot(1+t)^{-1} = \lambda_t[\mathscr{G}]\cdot(1-t+t^2-t^3+\ldots). \tag{5.98}$$

Comparing terms of equal degree in t, we have (1). To prove (2), let $\mathscr{L}$ be an invertible sheaf on Y. Then

$$\lambda^i([\mathscr{G}]\cdot[\mathscr{L}]) = [\mathscr{L}]^i \cdot \lambda^i[\mathscr{G}] \Longrightarrow \lambda^i([\mathscr{G}]\cdot[\mathscr{L}]) \equiv \lambda^i[\mathscr{G}] \mod (1-[\mathscr{L}]). \tag{5.99}$$

Then we have (2). In particular, if $\mathscr{G}_1, \mathscr{G}_2$ are both locally free sheaves on Y, the relation $[\mathscr{G}_1] \equiv [\mathscr{G}_2] \mod (1-[\mathscr{L}])$ implies that $\lambda^i[\mathscr{G}_1] \equiv \lambda^i[\mathscr{G}_2] \mod (1-[\mathscr{L}])$ for $i \geq 1$.

We now use these in our proof of the lemma. Applying (1) to $\mathscr{F}^*$,

$$(-1)^{p-1}\lambda_{-1}[\mathscr{F}^*] = \lambda^{p-1}([\mathscr{F}^*]-1). \tag{5.100}$$

We have that $\mathscr{N}^*/\mathscr{F}^* = \mathscr{L}^*$, so $[\mathscr{F}^*]-1 \equiv [\mathscr{N}'^*]-2 \mod (1-[\mathscr{L}^*])$, and (2) then implies that

$$(-1)^{p-1}\lambda_{-1}[\mathscr{F}^*] \equiv \lambda^{p-1}([\mathscr{N}'^*]-2) \mod (1-[\mathscr{L}^*]). \tag{5.101}$$

Finally, since $\lambda^{p-1}([\mathscr{N}'^*]-2) = g^*\lambda^{p-1}([\mathscr{N}^*]-2)$, it suffices to show that $\lambda^{p-1}([\mathscr{N}^*]-2) \equiv 0 \mod (1-[\mathscr{L}^*])$. This follows directly from the above lemma. □

Finally, we are in a position to prove:

Proposition 5.30. *GRR is true for a closed immersion $i: Y \to X$ where X, Y are non-singular, quasi-projective varieties.*

Proof. By Corollary 5.23 and the preceding results, it suffices to prove

$$\mathrm{ch}(i_*(y)) = i_*(\mathrm{ch}(y)\cdot \mathrm{td}(\mathscr{N})^{-1}) \tag{5.102}$$

when $y \in K(Y)$ and $p \geq \dim(Y)+2$. By Lemma 5.28, we have that $g^*(y)\cdot\lambda_{-1}[F^*] \equiv \mod (1-[\mathscr{L}^*])$. By Proposition 5.19, this lies in the inverse image $j^*(K(X'))$, so we apply the divisor case of GRR to $j: Y' \to X'$. Thus,

$$\mathrm{ch}(j_*(g^*(y)\cdot\lambda_{-1}[\mathscr{F}^*])) = j_*(\mathrm{ch}(g^*(y)\cdot\lambda_{-1}[\mathscr{F}^*])\cdot \mathrm{td}(\mathscr{L})^{-1}). \tag{5.103}$$

Now, in order to prove (5.104), we establish the following equalities:

1. $f_*(\mathrm{ch}(j_*(g^*(y)\cdot\lambda_{-1}[\mathscr{F}^*]))) = \mathrm{ch}(i_*(y))$
2. $f_*(j_*(\mathrm{ch}(g^*(y)\cdot\lambda_{-1}[\mathscr{F}^*])\cdot\mathrm{td}(\mathscr{L})^{-1})) = i_*(\mathrm{ch}(y)\cdot\mathrm{td}(\mathscr{N})^{-1})$

By Lemma 5.27, we observe that the left-hand side of (1) is equal to

$$f_*(\mathrm{ch}(f^*i_*(y))) = f_*f^*(\mathrm{ch}(i_*(y))) = \mathrm{ch}(i_*(y)), \tag{5.104}$$

where the last equality follows from Lemma 5.25. Proving (2) requires a little more work. First, we have

$$\mathrm{ch}(g^*(y)\cdot\lambda_{-1}[\mathscr{F}^*]) = \mathrm{ch}(g^*(y))\cdot\mathrm{ch}(\lambda_{-1}[\mathscr{F}^*]) = g^*(\mathrm{ch}(y))\cdot\mathrm{ch}(\lambda_{-1}[\mathscr{F}^*]), \tag{5.105}$$

which, by Lemma 5.24, is equal to

$$g^*(\mathrm{ch}(y))\cdot c_{p-1}(\mathscr{F})\cdot\mathrm{td}(\mathscr{F})^{-1}. \tag{5.106}$$

Furthermore, $\mathscr{N}'/\mathscr{L} = \mathscr{F}$ implies that

$$g^*(\mathrm{td}(\mathscr{N})) = \mathrm{td}(\mathscr{N}') = \mathrm{td}(\mathscr{N}) \cdot \mathrm{td}(\mathscr{L}). \tag{5.107}$$

We apply this to (5.106):

$$\mathrm{ch}(g^* \cdot \lambda_{-1}[\mathscr{F}^*] \cdot \mathrm{td}(\mathscr{L})^{-1}) = c_{p-1}(\mathscr{F}) \cdot g^*(\mathrm{ch}(y) \cdot \mathrm{td}(\mathscr{N})^{-1}). \tag{5.108}$$

Now apply the proper push forward g_* to the left-hand side and invoke Lemma 5.28

$$g_*(\mathrm{ch}(g^*(y) \cdot \lambda_{-1}[\mathscr{F}^*] \cdot \mathrm{td}(\mathscr{L})^{-1}) = \mathrm{ch}(y) \cdot \mathrm{td}(\mathscr{N})^{-1}. \tag{5.109}$$

Finally, we return to our commutative blow-up diagram, which yields $f_*j_* = i_*g_*$, and apply i_* to obtain

$$f_*j_*(\mathrm{ch}(g^*(y) \cdot \lambda_{-1}[\mathscr{F}^*]) \cdot \mathrm{td}(\mathscr{L})^{-1}) = i_*(\mathrm{ch}(y) \cdot \mathrm{td}(\mathscr{N})^{-1}), \tag{5.110}$$

which gives (2), and therefore proves the proposition. □

Recalling the reduction of Theorem 5.1 (GRR) to the following components:

1. Establish that $f : X \times \mathbb{P}^n \to X$ is projection onto the first factor,
2. Show that $f : Y \to X$ is an immersion onto a closed subvariety,

which were respectively established in Corollary 5.11 and Proposition 5.30, the Grothendieck-Riemann-Roch theorem for non-singular, quasi-projective varieties is finished.

5.5 Applications

In this brief subsection, we continue to follow [1]; however, we provide further applications later in this paper. Of course, we would be remiss to not include the necessary specialization of Theorem 5.1:

Corollary 5.31. *The Grothendieck-Riemann-Roch Theorem implies the Hirzebruch-Riemann-Roch Theorem: for any vector bundle E on a smooth, projective n-dimensional variety X, we have that*

$$\chi(X, E) = \int_X \mathrm{ch}(E) \cdot \mathrm{td}(T_X). \tag{5.111}$$

In particular, $\chi(X, \mathscr{O}_X) = \deg(\mathrm{td}(T_X))$.

Proof. We have already suggested the method of proof in the introduction. Let $\mathscr{E}$ be the locally free sheaf corresponding to E. Consider $f : X \to Y$ and $Y = \{\text{point}\}$ and apply the Grothendieck-Riemann-Roch theorem. In this case, the pushforward f_* on the Chow ring is simply the degree map $\int_X$. Now we need only show that $\mathrm{ch}(f_*[\mathscr{E}]) = \chi(X, E)$. We know that $K(\text{point}) \cong \mathbb{Z}$. Then

$$f_*[E] = \sum_{i \geq 0} (-1)^i [R^i f_*(\mathscr{E})] = \sum_{i \geq 0} (-1)^i [\widetilde{H^i(X, \mathscr{E})}], \tag{5.112}$$

by Propositions 4.15. The Chern character takes the rank on $Y = \{\text{point}\}$. Thus,

$$\mathrm{ch}(f_*[\mathscr{E}]) = \sum_{i \geq 0} (-1)^i \mathrm{rank}(H^i(X, \mathscr{E})) = \chi(X, E). \tag{5.113}$$

□

The following application is due to Hirzebruch, regarding integration on a complex algebraic fiber bundle.

Proposition 5.32. *Let $p : E \to B$ be a complex algebraic fiber bundle with fiber F. Then*

$$p_*(\mathrm{td}(T_F)) = g_{todd}(F) \cdot 1, \tag{5.114}$$

where $g_{todd}(X)$ is the Todd genus of a variety X, defined to be the degree of $\mathrm{td}(T_X)$.

Proof. Consider the short exact sequence

$$0 \longrightarrow T_F \longrightarrow T_E \longrightarrow p^* T_B \longrightarrow 0\ . \tag{5.115}$$

Using this and the naturally of the Todd class, we have $\mathrm{td}(T_E) = p^* \mathrm{td}(T_B) \cdot \mathrm{td}(T_F)$. Then

$$p_* \mathrm{td}(T_E) = \mathrm{td}(T_B) \cdot p_* \mathrm{td}(T_F), \tag{5.116}$$

by the Projection formula. Now apply GRR to p and the trivial fiber $1 \in K(X)$ on X:

$$p_*(\mathrm{td}(T_X)) = \mathrm{td}(T_B) \cdot \mathrm{ch}(p_*(1)). \tag{5.117}$$

From (5.110), we get

$$\mathrm{ch}(p_*(1)) = p_*(\mathrm{td}(T_F)). \tag{5.118}$$

Now we wish to compute $p_*(1)$. Let $U \subset B$ be an open affine for which $p : X \to U$ is trivial. Then, $H^i(U, \mathscr{O}_U) = 0$ for $i > 1$. Invoking the Künneth formula,

$$H^q(U \times F, \mathscr{O}_X) = \sum_{i+j=q} H^i(U, \mathscr{O}_U) \otimes H^j(F, \mathscr{O}_F) = \mathscr{O}_U(U) \otimes H^k(F, \mathscr{O}_F). \tag{5.119}$$

The structure group G of our fiber bundle $p : E \to B$ acts trivially on $H^k(F, \mathscr{O}_F)$. Hence,

$$p_*(1) = [\mathscr{O}_B] \cdot \left(\sum_{k \geq 0} (-1)^k [H^k(F, \mathscr{O}_F)] \right). \tag{5.120}$$

Since $p_*(1)$ is a linear combination of the classes of trivial vector bundles,

$$\mathrm{ch}(p_*(1)) = \sum_k (-1)^k \dim H^k(F, \mathscr{O}_F) \subset A_0(B). \tag{5.121}$$

Hence, $g_{\mathrm{todd}}(F) = \mathrm{td}(T_F) \cdot [F] = \mathrm{ch}(p_*(1))$. □

6 Riemann-Roch Algebra

In this section we follow the approach of [3] in formulating "Riemann-Roch functors," i.e., covariant-contravariant pairs satisfying particular axioms with a natural transformation between the contravariant parts. In this approach Riemann-Roch type theorems can be proved in an entirely algebraic setting, independent of underlying geometric considerations.

6.1 A General Situation

Consider two contravariant functors K and A from a category to the category of rings, and a natural transformation $\varphi : K \to A$. Since the Chern character considered in §5 is the most important example of this transformation, any such homomorphism

$$\varphi_X : K(X) \longrightarrow A(X) \tag{6.1}$$

is simply called a character.

For some morphism $f : X \to Y$, we have pullback homomorphisms:

$$f^K : K(Y) \to K(X), \qquad f^A : A(Y) \to A(X). \tag{6.2}$$

When acting as functors to abelian groups, K and A also exhibit covariant behavior with pushforward homomorphisms

$$f_K : K(X) \to K(Y), \qquad f_A : A(X) \to A(Y). \tag{6.3}$$

In general, these pushforwards do not commute with the character. However, we can construct an element $\tau_f \in A(X)$ such that the following commutes:

$$\begin{array}{ccc} K(X) & \xrightarrow{\tau_f \cdot \varphi_X} & A(X) \\ {\scriptstyle f_K}\downarrow & & \downarrow{\scriptstyle f_A} \\ K(Y) & \xrightarrow[\varphi_Y]{} & A(Y) \end{array} \tag{6.4}$$

When this diagram is commutative a Riemann-Roch type theorem is said to hold. We have already noted a common thread in Riemann-Roch theorems: for a morphism $f : X \to Y$ under consideration, we have a factorization:

$$X \xrightarrow{i} \mathbb{P}^n \xrightarrow{p} Y\ , \tag{6.5}$$

where i is a closed embedding and p the projection map. From this property we will construct an abstract formalism of Riemann-Roch functors that allows us to deduce very general Riemann-Roch theorems from the simple case of i an elementary embedding and p a bundle projection.

6.2 Riemann-Roch Functors

Consider a category $\mathscr{C}$ and simultaneously contra- and covariant functors H on $\mathscr{C}$. For each $X \in \text{ob}(\mathscr{C})$, H associates a ring $H(X)$, and for each morphism $f : X \to Y$, homomorphisms

$$f^H : H(Y) \to H(X), \qquad f_H : H(X) \to H(Y). \tag{6.6}$$

These homomorphisms satisfy the following axioms:

A(1): $X \mapsto H(X)$ is a contravariant functor from $\mathscr{C}$ to rings via f^H.

A(2): $X \mapsto H(X)$ is a covariant functor from $\mathscr{C}$ to abelian groups via f_H.

A(3): For all morphisms $f : X \to Y$ and all $x \in H(X), y \in H(Y)$, we have a natural projection formula:

$$f_H(x \cdot f^H(y)) = f_H(x) \cdot y. \tag{6.7}$$

In particular, $f_H(f^H(y)) = f_H(1)y$.

In our version of the Grothendieck-Riemann-Roch theorem, these homomorphisms were simply f^* and f_*.

Definition 6.1. A *Riemann-Roch functor* is a triple (K, φ, A), where K and A are functors satisfying axioms A(1)-A(3), and $\varphi : K \to A$ is a morphism of contravariant functors, i.e., where for each X, $\varphi_X : K(X) \to A(X)$ is a ring homomorphism, and

$$f^A \varphi_Y(y) = \varphi_X(f^K(y)) \tag{6.8}$$

for all $f : X \to Y, y \in K(Y)$.

We call φ the *Riemann-Roch character* (the Chern character in the previous section). We say that *Riemann-Roch holds* for a morphism f if, for some $\tau_f \in A(X)$, the following diagram is commutative:

$$\begin{array}{ccc} K(X) & \xrightarrow{\tau_f \cdot \varphi} & A(X) \\ {\scriptstyle f_K}\downarrow & & \downarrow{\scriptstyle f_A} \\ K(Y) & \xrightarrow{\varphi} & A(Y) \end{array} \tag{6.9}$$

Equivalently, for all $x \in A(X)$,

$$\varphi_Y f_K(x) = f_A(\tau_f \cdot \varphi_X(x)). \tag{6.10}$$

We call τ_f the *Riemann-Roch multiplier*. This element is the measure of the failure of φ to be covariantly functorial. Recall that in the case for non-singular quasi-projective varieties,

$$\tau_f = \mathrm{td}(T_X). \tag{6.11}$$

In particular, when Y is a divisor D on X: $\mathrm{td}(\mathscr{O}(D)) = [D] \cdot (1 - e^{-[D]})^{-1}$.

The general conditions for Riemann-Roch to hold are given by:

Theorem 6.2. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms. Assume that Riemann-Roch holds for f and g with associated multipliers τ_f and τ_g. Then Riemann-Roch holds for the composition $g \circ f$ with multiplier*

$$\tau_{g \circ f} = f^A(\tau_g) \cdot \tau_f. \tag{6.12}$$

Proof. Consider:

$$\begin{aligned} \varphi_Z(g_K f_K(x)) &= g_A(\tau_g \cdot \varphi_Y f_K(x)) && (6.13) \\ &= g_A(\tau_g \cdot f_A(\tau_f \cdot \varphi_X(x))) = g_A f_A(f^A(\tau_g) \cdot \tau_f \cdot \varphi_X(x)), && (6.14) \end{aligned}$$

where the equalities follow from Riemann-Roch for g, f, and the projection formula, respectively. □

This next theorem gives a means to determine Riemann-Roch multipliers for certain embeddings.

Theorem 6.3. *If $f^K : K(Y) \to K(X)$ is surjective, and there is an element $\tau \in A(Y)$ such that*

$$\varphi_Y(f_K(1)) = f_A(1)\tau, \tag{6.15}$$

then Riemann-Roch holds for f with multiplier

$$\tau_f = f^A(\tau). \tag{6.16}$$

Proof. For some $x \in K(X)$, let $x = f^K(y)$, where $y \in K(Y)$. Then compute

$$\varphi f_K(x) = \varphi f_K f^K(y) = \varphi(f_K(1)y) = \varphi(f_K(1))\varphi(y) = f_A(1)\tau\varphi(y) \tag{6.17}$$

$$= f_A(f^A(\tau\varphi(y))) = f_A(f^A(\tau)f^A\varphi(y)) = f_A(f^A(\tau)\varphi f^K(y)) = f_A(\tau_f\varphi(x)). \tag{6.18}$$

□

6.3 Chern Class Functors

We can specify Riemann-Roch functors to the case of Chern classes. A *Chern class functor* on $\mathscr{C}$ is a triple (K, c, A) where K and A are functors satisfying axioms A(1)-A(3) for each $X \in \mathscr{C}$ and c is a Chern class homomorphism:

$$c_X : K(X) \longrightarrow 1 + A(X)^+. \tag{6.19}$$

This homomorphism must satisfy the following axioms:

C(1): Each $K(X)$ is a λ-ring with involution, and f^K is a homomorphism of λ-rings.

C(2): Each $A(X)$ is a graded ring, and f^A is a graded ring homomorphism of degree 0.

C(3): For $f : X \to Y$ and $y \in K(Y)$, we have

$$f^A c(y) = c(f^K(y)). \tag{6.20}$$

We know that f^A and f^K are ring homomorphisms, so when A is a $\mathbb{Q}$-algebra, we have the functorial rules

$$f^A \operatorname{ch}(y) = \operatorname{ch}(f^K(y)), \qquad f^A \operatorname{td}(y) = \operatorname{td}(f^K(y)). \tag{6.21}$$

Thus, if $X \mapsto (K(X), c_X, A(X))$ is a Chern class functor, then

$$X \mapsto (K(X), \operatorname{ch}_X, A(X) \otimes_{\mathbb{Z}} \mathbb{Q}) \tag{6.22}$$

is a Riemann-Roch functor.

6.4 Elementary Embeddings and Projections

A morphism $f : X \to Y$ is an elementary embedding with respect to the Chern class functor (K, c, A) if

$$f^K : K(Y) \longrightarrow K(X) \tag{6.23}$$

is surjective, and there exists a positive element $q \in K(Y)$ (the principal element) such that

$$f_K(1) = \lambda_{-1}(q), \qquad f_A(1) = c_{\text{top}}(q^*). \tag{6.24}$$

The surjectivity condition on f^K holds when there is a morphism $p : Y \to X$ where $p \circ f = \operatorname{id}_X$.

We can explicitly determine our multiplier using the principal element. For $(K, \operatorname{ch}, A_{\mathbb{Q}})$ we have the following:

Theorem 6.4. *Riemann-Roch holds for elementary embeddings, with multiplier*

$$\tau_f = \mathrm{td}(f^K q^*)^{-1}. \tag{6.25}$$

Proof. This theorem follows from Theorem 6.3 and the fact that $\mathrm{td}(x)\,\mathrm{ch}(\lambda_{-1}(x^*)) = c_{\mathrm{top}}(x)$ (see Lemma 5.24) for some positive element x. □

We can also consider the "dual" situation where $f : X \to Y$ is an elementary projection, which imposes isomorphism conditions on $f_K : K(X) \to K(Y)$. We will not go into a full exposition of this, but again we can explicitly find the Riemann-Roch multiplier τ_f for this case.

This section tells us that, to prove Riemann-Roch for a morphism f with respect to (K, ch, A), it suffices to factor $f = p \circ i$, where p is an elementary projection and i is an elementary embedding (or admits a basic deformation to an elementary embedding).

7 Grothendieck's γ-Filtration

In this section we explore a celebrated consequence of the Grothendieck-Riemann-Roch theorem: the Chern character induces a multiplicative isomorphism

$$\mathrm{ch} : K_0(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} A^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}\,. \tag{7.1}$$

This can be generalized further; indeed, we consider the λ-ring structure of $K(X)$ and using Grothendieck's γ-filtration, construct a graded object $\mathrm{Gr}\, K(X)$ such that we have the following isomorphism:

$$\mathrm{ch} : K(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \mathrm{Gr}\, K(X) \otimes_{\mathbb{Z}} \mathbb{Q}\,. \tag{7.2}$$

This γ-filtration was developed by Grothendieck in order to replace the more natural "topological" filtration on $K_0(X)$. Let X be a Noetherian scheme. Then, the topological filtration is given by $F^nX/F^{n+1}X$ where F^nX is the subgroup of $K_0(X)$ generated by the classes of sheaves $\mathscr{F}$ for which $\mathrm{codim}(\mathrm{supp}(\mathscr{F})) > n$. There are, however, two issues with the topological filtration:

1. It can be applied to $K^0(X)$ only when $K_0(X) = K^0(X)$.
2. Even when we have this equality, we do not know if it is compatible with the ring structure of $K_0(X)$. An affirmative answer is only known for varieties over a field.

Grothendieck tackled these problems by defining the γ-filtration on $K^0(X)$. We shall describe this filtration, construct a new graded $K(X)$ associated to it, and finally prove that the Chern character is an isomorphism. This could be done in the algebraic context of the above section; however, we restrict ourselves to the more classical methods in [7].

Definition 7.1. There are operations $\gamma^i : K(X) \to K(X)$ for $i = 0, 1, 2, \ldots$ given by

$$\gamma_t(x) = \sum_{i=0}^{\infty} \gamma^i(x)t^i = \lambda_{\frac{t}{1-t}}(x) = \sum \lambda^i(x)(t + t^2 + \cdots) \in 1 + tK(X)[[t]]. \tag{7.3}$$

Such operations also satisfy:

P(1): $\gamma_t(x + y) = \gamma_t(x) \cdot \gamma_t(y)$;

P(2): $\gamma_t(1) = 1 + \frac{t}{1-t} = \frac{1}{1-t}$; $\gamma_t(-1) = 1 - t$;

P(3): Let ℓ be the class of some invertible sheaf $\mathscr{F}$. Then

$$\gamma_t(\ell - 1) = \gamma_t(\ell)\gamma_t(-1) = 1 + t(\ell - 1), \qquad \gamma_t(1 - \ell) = \frac{1}{\gamma_t(\ell - 1)} = \sum_{i=0}^{\infty} (1 - \ell)^i t^i. \tag{7.4}$$

Definition 7.2. There is a filtration on the ring $K(X)$ given by

$$F^1K(X) = \ker(\varepsilon : K(X) \to \mathbb{Z}), \tag{7.5}$$

where ε is the homomorphism that maps the class of a locally free sheaf on X to the rank of its stalk. Furthermore, we define $F^nK(X)$ to be the $\mathbb{Z}$-module generated by elements $\{\gamma_1^r x_1, \ldots, \gamma_k^{r_k} x_k\}$ where $x_i \in F^1K(X)$ and $\sum r_i \geq n$.

It is immediate that this is a filtration. Also, F^n is an ideal for all n since

$$x\gamma^{r_1}(x_1) \cdots \gamma^{r_k}(x_k) = (x - \varepsilon(x))\gamma^{r_1}(x_1) \cdots \gamma^{r_k}(x_k) + \epsilon(x)(\cdots), \tag{7.6}$$

and the first term is in F^{n+1}.

Proposition 7.3. *When the additive subgroup of the ring $K(X)$ is generated by classes of invertible sheaves on X*

$$F^iK(X) = (F^1K(X))^i. \tag{7.7}$$

Proof. Let ℓ_i be classes of invertible sheaves on X. We have that

$$(\ell_1 - 1)\cdots(\ell_i - 1) = \gamma^1(\ell_1 - 1)\cdots\gamma^i(\ell_i - 1), \tag{7.8}$$

so $(F^1K(X))^i \subset F^iK(X)$. Conversely, we need only prove that $\gamma^i(x) \in (F^1)^i$ for all $i \geq 1$ and $x \in K$. This is immediate by $\gamma^j(1-\ell)$ and $\gamma^j(\ell - 1)$, which were calculated in P(3). □

Example 7.4. Take $\ell = k(\mathscr{O}(1))$. Then $F^iK(\mathbb{P}^r_k) = ((\ell-1)^i)$.

Proposition 7.5. *Suppose there is an ample sheaf on X. Then $F^1K(X)$ is the nilradical of the ring $K(X)$.*

Proof. It follows from the Splitting Principle that we need only show that elements of the form $\ell - 1$ are nilpotent. We know there is an ample sheaf $\mathscr{O}_X(1)$ on X where we have a twisting operation, so for a sufficiently large integer n there is an integer m and an exact sequence

$$\mathscr{O}_X^m \longrightarrow \mathscr{L}^{-1}(n) \longrightarrow 0\ . \tag{7.9}$$

Thus,

$$\mathscr{L} \otimes \mathscr{O}_X^m(-n) \longrightarrow \mathscr{O}_X \longrightarrow 0. \tag{7.10}$$

Let $\ell_1 = [\mathscr{O}_X(1)]$. Then the kernel of the final homomorphism above is locally free with rank $m-1$. Thus, in $K(X)$, we have

$$0 = \lambda^m(m\ell\ell_1^{-n} - 1) = \sum_{i=0}^{m}(-1)^{m-i}\lambda^i(m\ell\ell_1^{-n}) \tag{7.11}$$

$$= (-1)^m\lambda_{-1}(m\ell\ell_1^{-n}) = (-1)^m(\lambda_{-m}(\ell\ell_1^{-n}))^m = (-1)^m(1-\ell\ell_1^{-n})^m. \tag{7.12}$$

Finally we note that

$$(1-\ell) = 1 - \ell\ell_1^{-n} + \ell(\ell_1^{-n} - 1), \tag{7.13}$$

and both summands are nilpotent if n is large enough. □

Corollary 7.6. *If $x \in F^1K(X)$, then $\gamma_t(x)$ is a polynomial.*

We now compute a specific example of this filtration:

Theorem 7.7. *Let $\mathscr{E} \in \mathfrak{Loc}(X)$ be of rank $r+1$, and let $x = [\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1))] - 1$. Then*

$$F^kK(\mathbb{P}(\mathscr{E})) = \sum_{i=0}^{r} F^{k-i}K(X)x^i. \tag{7.14}$$

Proof. We begin by showing that this equality holds for the infinite sum

$$\sum_{i=0}^{\infty} F^{k-i}K(X)x^i = \Omega^kK(\mathbb{P}(\mathscr{E})). \tag{7.15}$$

Ω^k is a filtration of the ring $K(\mathbb{P}(\mathscr{E}))$ and $\Omega^k(K(\mathbb{P}(\mathscr{E})) \subset F^k K(\mathbb{P}(\mathscr{E}))$. We must prove the reverse inclusion. It suffices to show that

$$\gamma^k(y) \in \Omega^k K(\mathbb{P}(\mathscr{E})) \tag{7.16}$$

for all $k \geq 1$ and all y in a set of generators for $F^1 K(\mathbb{P}(\mathscr{E}))$.

For this system of generators we choose elements $\alpha(\ell^m - 1)$ where $\alpha \in K(X), \ell = x+1, m \geq 1$, and $\beta \in F^1 K(X)$.

It is clear that $\gamma^k(K(X)) \subset \Omega^k K(\mathbb{P}(\mathscr{E}))$. Then we must show

$$\gamma^k(\alpha(\ell^m - 1)) \subset \Omega^k(K(\mathbb{P}(\mathscr{E})). \tag{7.17}$$

Let ℓ_1, ℓ_2 be the classes of two invertible sheaves $\mathscr{F}_1, \mathscr{F}_2$ on a scheme. Set $x_1 = \ell_1 - 1$, $x_2 = \ell_2 - 1$ and $x_{1,2} = \ell_1\ell_2 - 1$. Then, by P(3) and the fact that $x_{1,2} = x_1x_2 + x_1 + x_2$, we have

$$\gamma^k(x_1x_2) = \gamma^k(x_{1,2} - x_1 - x_2) = \sum_{p+q+r=k} \gamma^p x_{1,2}\gamma^q(-x_1)\gamma^r(-x_2) \tag{7.18}$$

$$= \sum_{q+r=k} (-x_1)^q(-x_2)^r + (x_1x_2 + x_1 + x_2) \sum_{q+r=k-1} (-x_1)^q(-x_2)^r = \sum_{i=0}^{\infty} P_{i,k}(x_1)x_2^i, \tag{7.19}$$

where the non-zero lowest term of the polynomial $P_{i,k}$ has deg $\geq k-i$. Now set $\alpha = \sum_j (\ell_1^j - 1)$. Letting $x_1^j - \ell_1^j - 1$, by P(1) we have

$$\gamma^k(\alpha x_2) = \sum_{i=0}^{\infty} Q_i(x_1^j)x_2^i, \tag{7.20}$$

where Q_i is the symmetric polynomial in the x_1^j such that the lowest non-zero homogeneous component has degree $\geq k-i$. Let S_n be the n-th elementary symmetric function in the x_i^j. Weight S_n by n to see that $Q_i x_1^j$ can be written as a polynomial $R_i(S_n)$ where each non-zero monomial has weight $\geq k-i$. By P(1), $S_n = \gamma^n(\alpha)$, so

$$\gamma^k(\alpha(\ell^m - 1)) = \sum_{i=0}^{\infty} R_i(\gamma^1\alpha, \cdots, \gamma^k\alpha,]\cdots)(\ell^m - 1)^i, \tag{7.21}$$

where $R_i(\gamma^1\alpha, \cdots, \gamma^k\alpha, \cdots) \in F^{k-i}K(X)$. By the Splitting Principle, this is true for arbitrary $\alpha \in K(X)$ that represent classes of $\mathscr{F} - \mathrm{rk}(\mathscr{F})$ with $\mathscr{F} \in \mathfrak{Loc}(X)$. These α generate the entire additive group $K(X)$. Finally, noting that

$$(\ell^m - 1)^i = x^i(\ell^{m-1} + \cdots + 1)^i = x^i f(x), \tag{7.22}$$

where f is a polynomial with integer coefficients, we get

$$F^k K(\mathbb{P}(\mathscr{E})) = \sum_{i=0}^{\infty} F^{k-i}K(X)x^i. \tag{7.23}$$

Now to show that this infinite sum is equal to $\sum_{i=0}^{r} F^{k-i}K(X)x^i$, we use a polynomial identity for x over $K(X)$:

Lemma 7.8. *For $e = [\mathscr{E}]$ and $r+1$ the rank of $\mathscr{E}$, we have that*

$$\sum_{i=0}^{r+1} (-1)^i \gamma^i(e - r - 1)x^{r+1-i} = 0. \tag{7.24}$$

Proof. We have that

$$\gamma_t(e-r-1) = \frac{\gamma_t(e)}{\gamma_t(r+1)} = (1-t)^{r+1}\lambda_{\frac{t}{1-t}}(e) = \sum_{i=0}^{r+1} \lambda^i(e)t^i(1-t)^{r+1-i}, \tag{7.25}$$

and so $\gamma_t(e-r-1)$ is a polynomial of degree $\leq r+1$. Thus,

$$\sum_{i=0}^{r+1} \gamma^i(e-r-1)t^{r+1-i} = t^r\gamma_{1/t}(e-r-1) = \sum_{i=0}^{r+1} \lambda^i(e)(t-1)^{r+1-i}. \tag{7.26}$$

Setting $t = 1-\ell = -x$, we have

$$\sum_{i=0}^{r+1} \gamma^i(e-r-1)(-1)^i x^{r+1-i} = \sum_{i=0}^{r+1} \lambda^i(e)(-1)^{r+1-i}\ell^{r+1-i} = 0, \tag{7.27}$$

which proves the lemma. ☐

Thus, for all $k \geq r+1$, we have

$$x^k \in \sum_{i=0}^{k-1} F^{k-i}K(X)x^i. \tag{7.28}$$

Inducting on $k-r$:

$$x^k \in \sum_{i=0}^{r} F^{k-i}K(X)x^i, \tag{7.29}$$

and we are done. ☐

Corollary 7.9. *In the same situation as above:*

$$f_*(F^kK(\mathbb{P}(\mathscr{E}))) \subset F^{k-r}K(X). \tag{7.30}$$

Proof. By the Projection formula

$$f_*(f^*y \cdot x^i) = f_*(x^j) \cdot y, \tag{7.31}$$

we have

$$f_*(F^iK(X) \cdot x^j) \in F^iK(X). \tag{7.32}$$

Invoking the previous theorem, we have our result. ☐

7.1 The Chern character isomorphism

We can associate a graded ring to the γ-filtration defined in the previous subsection:

$$\mathrm{Gr}K(X) = \bigoplus_{k=0}^{\infty} F^kK(X)/F^{k+1}K(X). \tag{7.33}$$

We write Gr^i for the i-th graded component of F^k/F^{k+1}.

Our goal will be the following theorem:

Theorem 7.10. *The Chern character induces an isomorphism of* $K(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ *with* $\operatorname{Gr} K(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

In order to prove this result, we will define a generalized Chern character for this situation, show that it is a ring homomorphism, and finally construct its inverse.

Definition 7.11. Let $\mathscr{E} \in \mathfrak{Loc}(X)$. Then

$$c_i(\mathscr{E}) = \gamma^i([\mathscr{E}] - \operatorname{rk}(\mathscr{E})) \mod F^{i+1}K(X) \in \operatorname{Gr}^i K(X) \tag{7.34}$$

is the i-th *Chern class* of $\mathscr{E}$ for $i \geq 1$.

This definition of the Chern class inherits all necessary properties of a characteristic class:

CC(1): For an invertible sheaf $\mathscr{L}$ on a scheme X, we have

$$c_i(\mathscr{L}) = \begin{cases} [\mathscr{L}] - 1 \mod F^2K(X), & i = 1 \\ 0 & i > 1. \end{cases} \tag{7.35}$$

CC(2): For any morphism $f : Y \to X$ of schemes, we have

$$c_i(f^*(\mathscr{E})) = \operatorname{Gr} f^*(c_i(\mathscr{E})), \tag{7.36}$$

where $\operatorname{Gr} f^* : \operatorname{Gr} K(X) \to \operatorname{Gr} K(Y)$ is induced by the ring homomorphism f^* compatible with the γ-filtration.

CC(3): Setting $c_t(\mathscr{E}) = 1 + \sum_{i=1}^{\infty} c_i(\mathscr{E}) t^i$, we have

$$c_t(\mathscr{E}) = c_1(\mathscr{E}') \cdot c_t(\mathscr{E}''), \tag{7.37}$$

for any short exact sequence $0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$.

CC(4): We have that

$$\operatorname{Gr} K(\mathbb{P}(\mathscr{E})) = \operatorname{Gr} K(X)[x] \tag{7.38}$$

where $x = [\mathscr{O}_{\mathbb{P}(\mathscr{E})}] - 1 \mod F^2K(\mathbb{P}(\mathscr{E}))$ satisfies

$$\sum_{i=0}^{r+1} (-1)^i c_i(\mathscr{E}) x^i = 0. \tag{7.39}$$

$$\tag{7.40}$$

CC(5): When $i > \operatorname{rk}(\mathscr{E})$, $c_i(\mathscr{E}) = 0$. Also, the mapping $\mathscr{E} \mapsto c_t(\mathscr{E})$ can b extended to a group homomorphism

$$c_t : K(X) \longrightarrow 1 + \bigoplus_{i=1}^{\infty} \operatorname{Gr}^i K(X) t^i. \tag{7.41}$$

Note that all of these properties are simply reformulations of Grothendieck's axiomatization of characteristic classes.

Definition 7.12. The Chern character of a locally free sheaf $\mathscr{E}$ on a scheme X is the element

$$\operatorname{ch}(\mathscr{E}) \in \operatorname{Gr} K(X) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{7.42}$$

given by

$$\operatorname{ch}(\mathscr{E}) = \sum \exp(\alpha_i(\mathscr{E})), \tag{7.43}$$

where the $\alpha_i(\mathscr{E})$ come from the identity

$$c_t(\mathscr{E}) = \prod_{i=1}^{\operatorname{rk}(\mathscr{E})} (1 + \alpha_i(\mathscr{E})t). \tag{7.44}$$

Proposition 7.13. *There is a ring homomorphism*

$$\operatorname{ch} : K(X) \longrightarrow \operatorname{Gr} K(X) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{7.45}$$

uniquely determined by $\operatorname{ch}([\mathscr{E}]) = \operatorname{ch}(\mathscr{E})$ *for all locally free sheaves* $\mathscr{E}$ *of finite rank on* X.

Proof. From CC(3), for any exact sequence of locally free sheaves $0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$, we have

$$\operatorname{ch}(\mathscr{E}) = \operatorname{ch}(\mathscr{E}') + \operatorname{ch}(\mathscr{E}''). \tag{7.46}$$

By the Splitting principle and CC(1), we have

$$\prod_i (1 + \alpha_i(\mathscr{E}' \otimes \mathscr{E}'')t) = \prod_{i,j} (1 + (\alpha_i(\mathscr{E}') + \alpha_j(\mathscr{E}''))t, \tag{7.47}$$

so $\operatorname{ch}(\mathscr{E}' \otimes \mathscr{E}'') = \operatorname{ch}(\mathscr{E}') \cdot \operatorname{ch}(\mathscr{E}'')$. □

In order to prove that the Chern character is an isomorphism, we must construct its inverse: the Adams operation.

Definition 7.14. Consider a λ-ring K. The *Adams power series* and *Adams operations* $\psi^j : K \to K$ are given by

$$\psi_t(x) = \varepsilon(x) - t\frac{d}{dt} \log \lambda_{-t}(x) = \sum_{j=0}^{\infty} \psi^j(x)t^j \tag{7.48}$$

Proposition 7.15. *The Adams operations* ψ^j *satisfy:*

1. $\psi^j(\ell) = \ell^j$ *if* ℓ *is the class of an invertible sheaf.*

2. *For all* j, *the map* ψ^j *is a ring homomorphism.*

3. $\psi^i(\psi^j(x)) = \psi^{ij}(x)$ *for all* $x \in K$ *and all* i, j.

The properties are essentially immediate. However, the behavior of the Adams operations with respect to the γ-filtration is much less trivial.

Proposition 7.16. *Let* $j \geq 1$. *Let* $n \geq 0$ *be an integer. If* $x \in F^n$, *then*

$$\psi^j(x) - j^n x \in F^{n+1}K(X). \tag{7.49}$$

Thus, $\operatorname{Gr}^n K$ *is an eigenspace for* $\operatorname{Gr} \psi^j$ *with eigenvalue* j^n.

Proof. The result is trivial for $n = 0$. It suffices to prove the result for elements $x = \gamma^n y$ where y runs through an additive basis of the additive group X. We take classes of locally free sheaves for y and apply the Splitting principle and Theorem 7.7, in order to reduce to the case where

$$x = \prod_{k=1}^{n} (\ell_k - 1) \tag{7.50}$$

and as usual ℓ_i are classes of invertible sheaves. Then

$$\psi^j(x) = \prod_{k=1}^{n} (\ell_k - 1) = \prod_{j=1}^{n} (\ell_k - 1) \prod_{k=1}^{n} (\ell_k^{j-1} + \cdots 1). \tag{7.51}$$

We see that

$$\ell_k^{j-1} + \cdots + 1 \equiv j \mod F^1 K(X). \tag{7.52}$$

Thus, $\psi^j(x) \equiv j^n x \mod F^{n+1} K(X)$. ☐

Corollary 7.17. *Let V_m be the subspace of $K(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ corresponding to the eigenvalue j^m of the Adam operator ψ^j for $j \geq 2$. Then, if $F^{d+1} K(X) = 0$ for some integer d, we have*

$$K(X) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{m=0}^{d} V_m \tag{7.53}$$

and the V_m is independent of j.

Proof. From the above proposition, it follows that

$$\prod_{n=0}^{d} (\psi^j - j^n) = 0 \tag{7.54}$$

as an operator of $K(X)$, and thus we can write the identity operator on $K(X) \otimes \mathbb{Q}$ as a decomposition of direct sum pairwise orthogonal projections

$$1 = \sum_{n=0}^{d} \prod_{m \neq n} (\psi^j - j^m)/(j^n - j^m). \tag{7.55}$$

Then the image of the m-th projection is just V_m.

To verify the independence from j, we write V_m as $V_{m,j}$. From the above proposition,

$$\prod_{m \neq n} (\psi^j - j^n)(\psi^m - k^m) = 0 \tag{7.56}$$

for any $k \in \mathbb{N}$. Thus, $V_{m,j} \subset V_{m,k}$ and so $V_{m,j} = V_{m.k} = V_m$ by symmetry. ☐

We now define a ring homomorphism

$$g : \operatorname{Gr} K(X) \otimes \mathbb{Q} \longrightarrow K(X) \otimes \mathbb{Q}\,. \tag{7.57}$$

For each non-zero $x \in \operatorname{Gr}^m K(X) \otimes \mathbb{Q}$ with $m \geq 1$, let $g(x) \in K(X) \otimes \mathbb{Q}$ denote the element $g(x) \in F^m K(X) \otimes \mathbb{Q}$ such that

1. $x = g(x) \mod F^{m+1} K(X) \otimes \mathbb{Q}$

2. $\psi^p(g(x)) = p^m g(x),\ p \geq 2$

Note that g is a well-defined ring homomorphism because $g(x)$ is single-valued.

We can now prove out main result, Theorem 7.10, by showing that ch and g are ring homomorphisms inverse to one another:

Proof. We consider elements x in the subring $K(X)$ (resp. $\operatorname{Gr} K(X)$) generated by classes of invertible sheaves. Let $x = \ell - 1 \mod F^2K(X) \in \operatorname{Gr}^1 K(X)$ with ℓ the class of an invertible sheaf. Then

$$g(x) = \log(1 + (\ell - 1)) = \sum_{n=1}^{\infty} (-1)^n \frac{(\ell - 1)^n}{n}. \tag{7.58}$$

It is immediate that the right-hand side $\mod F^2$ is equal to $\ell - 1$ in $F^1/F^2 = \operatorname{Gr}^1$. Then, since ψ^j is a ring homomorphism, we can apply ψ^j term-wise to get the eigenspace property for the expression on the right-hand side.

Now, simply by definition,

$$\operatorname{ch}(\ell - 1) = e^x - 1 \in \operatorname{Gr} K(X) \otimes \mathbb{Q}, \tag{7.59}$$

and so

$$g \circ \operatorname{ch}(\ell - 1) = \ell - 1, \qquad \operatorname{ch} \circ g(x) = x \tag{7.60}$$

and we are done. □

A Lemmata for §4.4

Here we collect a few results needed to prove the equality of $K_0(X)$ and $K^0(X)$.

Proposition A.1. *Let $0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{F}'$ be an exact sequence of $\mathscr{O}_X$-modules with $\mathscr{F}, \mathscr{F}'$ locally free sheaves. Then $\mathscr{E}$ is locally free.*

Theorem A.2. *(Syzygy Theorem) Let A be a regular local ring of dimension n and M a finitely generated A-module. Then M has a resolution by free A-modules of length n.*

Lemma A.3. *Suppose that X is non-singular with* $\dim(X) = n$*. Let $\mathscr{F}$ be a coherent sheaf and suppose that*

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{G}_k \longrightarrow \cdots \longrightarrow \mathscr{G}_1 \longrightarrow \mathscr{G}_0 \longrightarrow \mathscr{F} \longrightarrow 0 \tag{A.1}$$

is an exact sequence of coherent sheaves, with $\mathscr{G}_i$ a locally free sheaf on X when $0 \leq i \leq k$. Then $\mathscr{K} \in \mathfrak{Loc}(X)$ whenever $k \geq n-1$.

Proof. First, note that $\mathscr{K}$ is coherent. We can localize the sequence and apply the Syzygy theorem to conclude that $\mathscr{K}_x$ is a finitely generated free $\mathscr{O}_{X,x}$-module when $k \geq \dim \mathscr{O}_{X,x} - 1$. Finally, observe that $n - 1 \geq \operatorname{codim}(\overline{\{x\}}, X) - 1 = \dim \mathscr{O}_{X,x} - 1$ for all $x \in X$. □

Corollary A.4. *Let X be a non-singular quasi-projective variety and let $\mathscr{F} \in \mathfrak{Coh}(X)$. Then there is a finite locally free resolution of $\mathscr{F}$. That is,*

$$0 \longrightarrow \mathscr{G}_n \longrightarrow \mathscr{G}_{n-1} \longrightarrow \cdots \mathscr{G}_0 \longrightarrow \mathscr{F} \longrightarrow 0 \tag{A.2}$$

with $\mathscr{G}_i \in \mathfrak{Loc}(X)$ for all i.

Lemma A.5. *Suppose that X is quasi-projective. Let $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathfrak{Coh}(X)$ and let $u : \mathscr{A} \to \mathscr{B}$ and $v : \mathscr{C} \to \mathscr{B}$ be surjective morphisms. Then there is a locally free $\mathscr{O}_X$-module $\mathscr{E}$ and morphisms $u' : \mathscr{E} \to \mathscr{A}$ and $v' : \mathscr{E} \to \mathscr{C}$ such that the compositions $u \circ v'$ and $v \circ u'$ are surjective:*

$$\begin{array}{ccc} \mathscr{E} & \xrightarrow{v'} & \mathscr{A} \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ \mathscr{C} & \xrightarrow[v]{} & \mathscr{B} \end{array} . \tag{A.3}$$

Proof. Let $\mathscr{D}$ be the sub sheaf of $\mathscr{A} \oplus \mathscr{C}$ of pairs (x, y) which have the same image in $\mathscr{B}$. Then the projections $\mathscr{D} \to \mathscr{A}$ and $\mathscr{D} \to \mathscr{C}$ are surjective, since u and v are, and form a commutative square. Furthermore, $\mathscr{D}$ is coherent. Now by Serre's theorem, write $\mathscr{D}$ as the quotient of a locally free sheaf $\mathscr{E} \to \mathscr{D}$, and define u' and v' by composing the projections with the quotient map. □

Lemma A.6. *Suppose that X is quasi-projective. Let $0 \to \mathscr{M} \to \mathscr{E} \to \mathscr{B} \to 0$ and $0 \to \mathscr{M}' \to \mathscr{E}' \to \mathscr{B}' \to 0$ be two exact sequences in $\mathfrak{Mod}(X)$ with $\mathscr{E}, \mathscr{E}' \in \mathfrak{Loc}(X)$. Suppose that $\mathscr{B}'' \in \mathfrak{Mod}(X)$ and we have surjections $\mathscr{B}'' \to \mathscr{B}$ and $\mathscr{B}'^{,\prime} \to \mathscr{B}'$. Then we have a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{M} & \longrightarrow & \mathscr{E} & \longrightarrow & \mathscr{B} & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathscr{M}'' & \longrightarrow & \mathscr{E}'' & \longrightarrow & \mathscr{B}'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{M}' & \longrightarrow & \mathscr{E}' & \longrightarrow & \mathscr{B}' & \longrightarrow & 0 \end{array} \tag{A.4}$$

where all rows are exact and all vertical morphisms are surjective.

Proof. This amounts to checking the requisite properties. See [1]. □

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