Math 101
Take Home Exam and Practice Final

To be done on your own. This should take about 3 hours. Versions of some problems here may appear on the final. Each is worth 10 points. Due in class on Tuesday, January 11.

1. Prove that if $f : B^2 \to B^2$ is continuous, and if $f(x) = x$ for all $x \in S^1$, then f is surjective. *Hint:* If f is not surjective then you can find a retraction from B^2 to S^1.

2. Let $P^2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1^2 + x_2^2 \leq 1\}$ denote the punctured unit disk in the plane.

   (a) Is P^2 closed? Complete? Compact? Justify your answers.

   (b) Give an example of a retraction $r : P^2 \to S^1$ where S^1 is the unit circle and prove that your choice of r has the properties required of a retraction.

   (c) Is the retraction r you found uniformly continuous on P^2? Is it a homeomorphism? Prove your assertions.

   (d) Can r extend to a continous function $f : B^2 \to S^1$ where B^2 denotes the closed unit disk and $f(x) = r(x)$ for $x \in P^2$? Prove your assertion.

3. Decide whether the following is true or false, then prove your assertion: A topological space (X, K) is connected if and only if there exists a retraction $r : X \to A$ onto a set A that is connected.

4. Decide whether the following is true of false, then prove your assertion: A topological space $(X, \mathbf{K})$ is compact if and only if every continuous function $f : X \to \mathbb{R}$ achieves its maximum on X.

5. (a) Show that every finite subset A of a metric space is closed with respect to the closure operator induced by the metric.

   (b) Decide whether the following is true or false, then prove your assertion: If $(X, \mathbf{K})$ is a topological space, every finite subset $A \subset X$ is closed.

6. (a) A real number $x \in \mathbb{R}$ is called algebraic if it is the root of a polynomial with integer coefficients. Show that the set of all algebraic numbers is countable. Conclude that there must exist numbers that are not algebraic even though it is hard to produce specific examples.

   (b) Use the Binomial Formula and what you know about the complex exponential $e^{i\theta}$ to derive an expression for $\cos(4\theta)$ that is a polynomial in $\sin(\theta)$ and $\cos(\theta)$.

7. Show that a topological space with the fixed point property must be connected.

8. Let $X \subset \mathbb{R}^n$ be noncompact with respect to $\mathbf{K}_X$, the standard Euclidean closure on $\mathbb{R}^n$ restricted to X.

   (a) Show that setting

$$\mathbf{K}A = \begin{cases} \mathbf{K}_X A & \text{if } A \subset \mathbb{R}^n \text{ and } \mathbf{K}_X A \text{ is compact in } \mathbb{R}^n, \\ \mathbf{K}_X A \cup \{\infty\} & \text{otherwise.} \end{cases}$$

   defines a closure operator on $\hat{X} = X \cup \{\infty\}$.

   (b) The topological space $(\hat{X}, \mathbf{K})$ is called the one-point compactification of A. The space we have been calling $\mathbb{L}$ arises as the compactification of what space? Does the usual metric extend to one that induces $\mathbf{K}$ on $\hat{X}$? Explain briefly.

   (c) Show that if $(X, \mathbf{K}_X)$ and $(Y, \mathbf{K}_Y)$ are homeomorphic noncompact subsets of $\mathbb{R}^n$, then $\hat{X}$ and $\hat{Y}$ are also homeomorphic.

   (d) Explain briefly (without proof) a conjecture identifying a familiar space that is homeomorphic to $\hat{\mathbb{R}}^2$, the one point compactification of the plane.

9. (a) Suppose that $a_k \geq 0$ for all $k \in \mathbb{Z}^+$. Consider the sums

$$S_n = \sum_{k=1}^{n} a_k$$

   Give a condition on the set $\{S_n \mid n \in \mathbb{Z}^+\}$ that is necessary and sufficient for there to exist a number $S = lim_{n \to \infty} S_n$.

(b) Show that, for all $n \in \mathbb{Z}^+$, we have

$$\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}$$

(c) Does there exist one and only one number $S = \lim_{n\to\infty} S_n$ where

$$S_n = \sum_{k=1}^{n} \frac{1}{(2k+1)(2k+3)} = \frac{1}{(1)(3)} + \frac{1}{(3)(5)} + \frac{1}{(5)(7)} + ... + \frac{1}{(2n+1)(2n+3)}?$$

(d) Here are two different arguments for the value of S. First, we could say

$$\begin{aligned} S &= (1/1 - 2/3) + (2/3 - 3/5) + (3/5 - 4/7) + \dots \\ &= 1 - 2/3 + 2/3 - 3/5 + 3/5 - 4/7 + \dots \\ &= 1 \end{aligned}$$

since all terms after the first cancel. Or we could say

$$\begin{aligned} S &= (1/1 - 1/3)/2 + (1/3 - 1/5)/2 + (1/5 - 1/7)/2 + (1/5 - 1/7)/2 + \dots \\ &= 1/2 - 1/6 + 1/6 - 1/10 + 1/10 - 1/14 + \dots \\ &= 1/2 \end{aligned}$$

since all terms after the first cancel. What is going on here? Can you rearrange terms in a sum or not? What is S?

10. Pick an interesting problem from the first exam that you could have done better and write up your best solution.