SOLUTIONS SET 12

THE LAST 101 SOLUTION SET YOU WILL EVER SEE

7 *Suppose* (X, d) *is a metric space topologized in the standard way and let* $B \subset X$*. Show that* $b \in KB$ *if and only if there is a sequence* $f\colon \mathbb{Z}^+ \to B$ *such that* $lim_{n\to\infty} f(n) = b$*.*

Suppose $b \in KB$. For every integer n, we can find an $f(n) \in B$ such that $d(b, x_n) < 1/n$. In this way define the function $f(n)$ on the positive integers, and furthermore $f(\infty) = b$. We need to show that f is continuous at ∞. Let $\varepsilon > 0$. Then of course we can find a positive integer N such that $N > 1/\epsilon$. If $n \geq N$, then $d(f(\infty), f(n)) = d(b, x_n) < 1/n \leq 1/N < \varepsilon$. It follows that indeed, f is continuous at infinity, therefore by definition $lim_{n\to\infty} f(n) = b$.

Conversely, suppose such an f exists. Let's show $b \in KB$. To do this, we begin by letting $\varepsilon > 0$. Since $lim_{n\to\infty} f(n) = b$, by definition the function

$$f(n) = \begin{cases} x_n & \text{if } n \in \mathbb{Z}^+ \\ b & \text{if } n = \infty \end{cases}$$

is continuous at ∞. Therefore if we let $\varepsilon > 0$, we can find an $n \in \mathbb{Z}^+$ such that $d(b, f(n)) < \varepsilon$. But $f(n) \in B$, so by definition $b \in KB$.

9 *Proof or counterexample: According to the definition we gave, must the limit of a sequence be unique? Does it matter whether or not there is a metric around?*

I didn't expect *anyone* to get the point of this, it's just too subtle. Way too subtle. Anyway, if the space in question has a metric topology, then yes, the limit is unique. But you can indeed define a limit without a metric (read the notes carefully!), and you can make up a sequence in a space with two limits. For instance, let X be the space $\mathbb{Z}^+ \cup \{\infty_1, \infty_2\}$ and define a closure operator on X via

$$KA = \begin{cases} A & \text{if } A \text{ is finite} \\ A \cup \{\infty_1, \infty_2\} & \text{if } A \text{ is infinite} \end{cases}.$$

Now consider the sequence $1, 2, 3, \ldots$ in X. Then it is not hard to show (try it!) that $f\colon \mathbb{L} \to X$ defined by

$$f(n) = \begin{cases} n & \text{if } n \in \mathbb{Z}^+ \\ \infty_1 & \text{if } n = \infty \end{cases}$$

is continuous; therefore by definition $\lim_{n\to\infty} n = \infty_1$. But by symmetry $\lim_{n\to\infty} n = \infty_2$.

If a topological space only admits one limit per sequence, we call it a Hausdorff space. That is, I suppose, what this problem is getting at, that distinction, and the fact that metric spaces are always Hausdorff. But again, I would be extremely impressed if anyone came up with a counterexample.

10 *Are uniform functions necessarily continuous?*

Yes, most of you got this one.

Must a continuous function be uniformly continuous?

No, the function $f(x) = x^2$ from $\mathbb{R}$ to $\mathbb{R}$ is a counterexample.

How would you describe a function $f\colon \mathbb{R} \to \mathbb{R}$ that satisfies the condition, which I will write in English because I haven't the patience to look up all the quantifier symbols: for all $x \in \mathbb{R}$, there exists a $\delta > 0$ such that for all $\varepsilon > 0$ $fB(x,\delta) \subset B(f(x),\varepsilon)$?

I'll prove in general that if $f\colon X \to Y$ is a function between two metric spaces satisfying the above criterion, then for all $y \in Y$, $f^{-1}(y) \subset X$ is open. For let $x \in f^{-1}(y)$ (that is, $f(x) = y$); we will show that there's an open ball $B(x,\delta)$ contained entirely in $f^{-1}(y)$. By the condition there's a $\delta > 0$ such that for all $\varepsilon > 0$, $fB(x,\delta) \subset B(f(x),\varepsilon)$. So if $a \in B(x,\delta)$, then $d(f(a), f(x)) < \varepsilon$ for all ε. But that means that $d(f(a), f(x)) = 0$, that is $f(a) = f(x) = y$ so $a \in f^{-1}(y)$. This is true for all $a \in B(x,\delta)$, so $B(x,\delta) \subset f^{-1}(y)$. This means $f^{-1}(y)$ is open.

But since f is continuous (or is it? yes, it is) and $\{y\}$ is closed, by the inverse mapping theorem $f^{-1}(y)$ is closed. So it is open and closed, for all y. It follows that the sets $f^{-1}(y)$ are all separated from one an other. The conclusion is that f is constant on all the connected components of X. (f is called locally constant.) In particular if X is connected, say $X = \mathbb{R}$, then f is constant.

11 *Show that if $f\colon \mathbb{Z}^+ \to Y$ for a metric space (Y, d_Y), then $\lim_{n\to\infty} = b$ in Y as we have defined this is equivalent to the usual definition.*

$\lim_{n\to\infty} = b$ means that function $f\colon \mathbb{L} \to Y$ defined by adding the value $f(\infty) = b$ is continuous at infinity. Suppose it is. Then for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $fB(\infty,\delta) \subset B(b,\varepsilon)$. But if we let $N > 1/\delta$, then the integers $n > N$ satisfy $1/n < 1/N < \delta$, that is $n \in B(\infty,\delta)$, so that $f(n) \in fB(\infty,\delta) \subset B(b,\varepsilon$, that is $d(f(n),\varepsilon) < \varepsilon$.

Conversely let b equal the limit in the usual sense. Then we want to show that the function $f\colon \mathbb{L} \to Y$ is continuous at infinity. If $\epsilon > 0$, let N be the point in the sequence where $n \geq N$ implies $d(f(n), b) < \varepsilon$; then for $\delta < 1/N$ we have $fB(\infty,\delta) \subset B(b,\varepsilon)$ as required.

12 *Show that $f\colon \mathbb{Z}^+ \to Y$ is uniformly continuous on $\mathbb{Z}^+$ if and only if for all $\varepsilon > 0$ there is an $N \in \mathbb{Z}^+$ such that $m, n > N$ implies $d(f(m), f(n)) < \varepsilon$.*

Say f is uniformly continuous. Let $\varepsilon > 0$. Then there's a $\delta > 0$ such that for all $m, n \in \mathbb{Z}^+$, $d(m, n) < \delta$ implies $d(f(m), f(n)) < \varepsilon$. Say $N > 2\delta$. Then for all $m, n > N$ we have $d(m, n) \leq d(m, \infty) + d(\infty, n) = 1/m + 1/n < 1/N + 1/N = 2/N < \delta$, implying $d(f(m), f(n)) < \varepsilon$ as required.

Conversely say that f satisfies the second condition. Let $\varepsilon > 0$. Then there is an $N \in \mathbb{Z}^+$ such that $m, n > N$ implies $d(f(m), f(n)) < \varepsilon$. Let $\delta < 1/N^2$. Then whenever $d(m, n) < \delta$, we have $1/N^2 > \delta > |1/m - 1/n| > |m - n|/mn \geq 1/mn \geq 1/\max m, n^2$, so that $\max m, n > N$, that is $m, n > N$, so that by the condition $d(f(m), f(n)) < \varepsilon$ as required.

None of you got this quite right, but many of you wrote something down that looked like you had the right idea but was very confusing with the variables. Again, this problem was very, (maybe too) hard.

Prove that a convergent sequence is Cauchy.

Most of you got this one.

Assuming $Y = \mathbb{R}$, prove that a Cauchy sequence converges.

Another of those way-too-hard problems. Say the sequence $f(n)$ is Cauchy. For every integer $p > 0$, let N_p be a point in the sequence for which $m, n \geq N_p$ imply $|f(m), f(n)| < 2^{-p-1}$. In particular, since $N_p, N_{p+1} \geq N_p$ it follows that $d(N_p, N_{p+1}) < 2^{-p-1}$. Let X_p be the closed interval $[f(N_p) - 2^{-p}, f(N_p) + 2^{-p}]$. We have that $f(N_{p+1}) - 2^{-(p+1)} > f(N_p) - 2^{-p-1} - 2^{-p-1} = f(N_p) - 2^{-p}$. Similarly $f(N_{p+1}) + 2^{-(p+1)} < f(N_p) + 2^{-p}$. Thus the X_p's are nested, implying there is an element x in their intersection. We claim $f(n)$ converges to x. For if $\varepsilon > 0$, let p be large enough so that $2^{-p+1} < \varepsilon/2$. Then if $n > N_p$, we have $|f(n), f(N_p)| < 2^{-p-1}$ by the Cauchy property. But then

$$\begin{aligned} |f(n), x| &\leq |f(n), f(N_p)| + |f(N_p), x| \\ &< 2^{-p-1} + |f(N_p), x| \\ &< 2^{-p-1} + 2^{-p+1} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

where the last step is because x and $f(N_p)$ both live in the interval X_p, which has length $2^{-p} + 2^{-p} = 2^{-p+1}$. Thus the $f(n)$ indeed converge to x.

We have used the fact that $\mathbb{R}$ is complete. In fact the property that every Cauchy sequence converges is equivalent to completeness. For example the rational numbers $\mathbb{Q}$ is not complete, and there are sequences (say $3, 3.1, 3.14, 3.141, \ldots$) that are Cauchy but do not converge to anything.

4 *Must f be continuous?*

No. If $f_n(x) = e^{-1/nx^2}$, with the added value $f_n(0) = 0$, then each f_n is continuous, but then the pointwise limit f is 0 at 0 and 1 everywhere else, so is not continuous.

5 *What speculations do you have about the continuity of power series?*

It happens to be the case that power series are continuous on their domain of convergence (that is, on the set of points at which the power series converges). This domain is always an interval. In fact, on the domain of convergence, the power series will be infinitely differentiable.