MATH 101 PROBLEM SET 2

SOLUTIONS

A. *Are the Kuratowaski Axioms independent of one another? Explain.*

What we were looking for here was not necessary a proof that the axioms are independent of each other, but rather a strategy for how this might be done. Many of you noted that each axiom addresses a different sort of set theoretic operation and listed this as evidence for their independence. This might have offered some intuition with the axioms, but it does not hold up in court, so to speak. Let's recall what it means for Axiom A to *not* be independent from Axioms B, C, etc. It means that assuming that $B, C, \ldots$ hold, then A follows as a theorem. But if we can construct a system which satisfies $B, C, \ldots$ but *not* A, then A cannot possibly follow from the others. Let's do this for each axiom.

Axiom C1: $A \subset \mathbf{K}A$. If X is any nonempty set and $\mathbf{K}$ is defined to have value $\emptyset$ on all subsets of X, then $(X, \mathbf{K})$ will satisfy C2, C3 and C4 but not C1.

Axiom C2: $\mathbf{K}(A \cup B) = \mathbf{K}A \cup \mathbf{K}B$. Let $X = \{1, 2, 3\}$, and define $\mathbf{K}$ as

$$\mathbf{K}A = \begin{cases} \emptyset & \text{if } A = \emptyset \\ X & \text{if } A = \{1, 3\} \\ A & \text{if } A \text{ is otherwise} \end{cases}$$

Axiom C3: $\mathbf{KK}A = \mathbf{K}A$. Let $X = \mathbf{Z}$ and let

$$\mathbf{K}A = A \cup (A + 1) = A \cup \{x + 1 \mid x \in A\}.$$

Then $\mathbf{KK}\{0\} = \{0, 1, 2\} \neq \{0, 1\} = \mathbf{K}\{0\}$. However, axioms C1, C2 and C4 all hold.

Axiom C4:. Simply let X be any nonempty set and let K have constant value X.

B. W. 5.3 The flaw here is, as most of you noted, the hasty phrase, "By the same reasoning, if $C = D$, then $A = B$. Theorem 4.7 simply does not apply in that direction. However, $A = B$ does indeed imply $C = D$.

Please note that all the following proofs, for all their terseness, would be perfectly acceptable if handed in as homework. I do not care if your proof is all English (as I prefer to write them), or all symbols. The point is that the reasoning is linear, which means that at every step I know how you can justify the proof but also why you are taking that step. Furthermore you should note that I will never require a justification for the most basic set-theoretic theorems. If it is obvious to you, it is probably obvious to me as well. (Perhaps this is a dangerous comment to put out there.) Well, here we go.

5.3 4) a) $\wp(A \cup B) = \wp(A) \cup \wp(B)$. The $\supset$ direction is true, because if $C \in \wp(A) \cup \wp(B)$ then $C \subset A$ or $C \subset B$, so in any case $C \subset A \cup B$, hence $C \in \wp(A \cup B)$. However, the $\subset$ direction is false. Let $A = \{1\}$ and $B = \{2\}$. Then $\{1,2\}$ is in $\wp(A \cup B)$ but neither in $\wp(A)$ nor $\wp(B)$.

b) $\wp(A - B) = \wp(A) - \wp(B)$. This is not true in either direction. For one thing, the $\subset$ direction is false because $\emptyset$ is always in the left hand side but never in the right hand side. In the other direction, let $A = \{1,2\}$ and $B = \{1\}$. Then $\{1,2\}$ is in $\wp(A) - \wp(B)$ but it is not in $\wp(A - B) = \wp(\{2\})$.

c) $\cup(\wp(A)) = A$. A word is in order about this. The notation on the left side means the union of all the sets in $\wp(A)$. This is true. For the $\subset$ direction, if $x \in \cup(\wp(A))$, then it must in a subset of A, hence it is in A itself. Conversely if $x \in A$ then x must also be in the union of any collection of sets of which A is a member. Thus $x \in \cup(\wp(A))$.

d) $\wp(\cup\mathcal{A}) = \mathcal{A}$. This is true only in the $\supset$ direction. If $A \in \mathcal{A}$, then A is naturally a subset of $\cup\mathcal{A}$ and hence a member of $\wp(\cup\mathcal{A})$. (Many of you confused the notions of "subset" and "member" here. A here is both a member of $\mathcal{A}$ and a subset of $\wp(\cup\mathcal{A})$.) As a counterexample for the reverse direction, let $\mathcal{A}$ consist of the single set $\{1\}$. Then $\emptyset \in \wp(\cup\mathcal{A})$ but not in $\mathcal{A}$ itself.

7,10. Almost all of you did these just fine.

16. You can only conclude a), "Each set in $\mathcal{A}$ is nonempty." The other statement, "Each set in $\mathcal{A}$ is disjoint," doesn't make sense; it's like saying that Central Square is equidistant from Harvard. And yet, the phrase "nonempty disjoint sets" wouldn't phase any mathematician. Indeed a large part of reading mathematics is divining the intent of the writer, who may often indulge himself in such "abuse of notation."

C. The correct generalization runs as follows. If $(X, \mathbf{K})$ is a topological spce, and A_i is a subset of X for $i = 1, 2, \ldots, n$, then

$$\mathbf{K}(\cup_{i=1}^{n} A_i) = \cup_{i=1}^{n} \mathbf{K} A_i.$$

We prove this using induction. The base case, $n = 1$, is completely trivial. Now assume the statement true for n; we shall prove it for $n + 1$. We have

$$K(\cup_{i=1}^{n+1} A_i) = K((\cup_{i=1}^{n} A_i) \cup A_{n+1}) = K(\cup_{i=1}^{n} A_i) \cup \mathbf{K} A_{n+1}$$

where we have just used Axiom C2. Now by the inductive hypothesis this last expression is

$$(\cup_{i=1}^{n} \mathbf{K} A_i) \cup \mathbf{K} A_{n+1} = \cup_{i=1}^{n} \mathbf{K} A_i$$

as required. Please note that the statement is false for $n = \infty$, despite its being true for arbitrarily large (but finite) n.

D. 3. The Int/Ext question. Many of you wrote up proofs which were novelettes. Others wrote up incomprehensible garbage. No one wants to be mired in a heap of algebraic statements in which it is never clear exactly which direction the proof is heading towards. Folks, I prize clarity over everything here. If you write up a clear proof, it is usually going to be correct. Hopefully the switching process will help in this department. Anyway, here we go.

a) Let $(X, \mathbf{K})$ be a topological space and let $A \subset X$. Show that each point of X is contained in exactly one of the sets $\operatorname{Int} A$, $\operatorname{Ext} A$, ∂A.

Proof. Let $x \in X$. We are concerned with the membership of x in the sets $\mathbf{K}A$ and $\mathbf{K}(X - A)$. There are three possibilities. The first is that $x \notin \mathbf{K}A$. The second is that $x \in \mathbf{K}A$ and also in $\mathbf{K}(X - A)$, and the third is that $x \in \mathbf{K}A$ but not in $\mathbf{K}(X - A)$. Certainly x falls into one of these categories, but no two of these possibilities can happen simultaneously. Let us examine each one.

$x \notin \mathbf{K}A$ if and only if $x \in X - \mathbf{K}A = \operatorname{Ext} A$.

$x \in \mathbf{K}A$ and $x \in \mathbf{K}(X - A)$ if and only if $x \in \mathbf{K}A \cap \mathbf{K}(X - A) = \partial A$.

$x \in \mathbf{K}A$ and $x \notin \mathbf{K}(X - A)$ implies that $x \in X - \mathbf{K}(X - A) = \operatorname{Int} A$. Conversely, if $x \in X - \mathbf{K}(X - A)$, then of course it cannot live in $X - A$, which means it must live in A, hence in $\mathbf{K}A$. We are done.

b) Show that $\mathbf{K}A = A \cup \partial A$.

Proof. Indeed, $A \cup \partial A = A \cup (\mathbf{K}A \cap \mathbf{K}(X - A)) = (A \cup \mathbf{K}A) \cap (A \cup \mathbf{K}(X - A)) = \mathbf{K}A \cap (A \cup \mathbf{K}(X - A))$. All we need to do now is prove that $\mathbf{K}A \subset (A \cup \mathbf{K}(X - A))$, so that this intersection will simply be $\mathbf{K}A$ as required. If $x \in \mathbf{K}A$, there are two possibilities. Either $x \in A$ itself, so that of course it must be in $A \cup \mathbf{K}(X - A)$, or $x \notin A$, so that $x \in X - A$ and hence $x \in \mathbf{K}(X - A)$, so again we have $x \in A \cup \mathbf{K}(X - A)$. That's all.

c) Show that $\operatorname{Int} A = A - \partial A$.

Proof. Suppose $x \in \operatorname{Int} A$. Then by 4a) we have $x \in A$, and furthermore since $\operatorname{Int} A$ and ∂A are disjoint we have $x \notin \partial A$. Thus $\operatorname{Int} A \subset A - \partial A$. Conversely, suppose $x \in A - \partial A$. Since it isn't in ∂A, by 3a) it must either be in $\operatorname{Int} A$ or $\operatorname{Ext} A$. But since $x \in A$ it is in $\mathbf{K}A$ and therefore is excluded from $\operatorname{Ext} A = X - \mathbf{K}A$. The only possibility is $x \in \operatorname{Int} A$.

4a) $\operatorname{Int} A \subset A$. If $x \in \operatorname{Int} A = X - \mathbf{K}(X - A)$, it is not in $\mathbf{K}(X - A)$, so in particular it is not in $X - A$, i.e. it is in A.

4b)

$$\begin{aligned}
\operatorname{Int}(A \cap B) &= X - \mathbf{K}(X - (A \cap B)) \\
&= X - \mathbf{K}((X - A) \cup (X - B)) \\
&= X - \mathbf{K}(X - A) \cap \mathbf{K}(X - B) \\
&= (X - \mathbf{K}(X - A)) \cap (X - \mathbf{K}(X - B)) \\
&= \operatorname{Int} A \cap \operatorname{Int} B
\end{aligned}$$

4c)

$$\begin{aligned}
\operatorname{Int}\operatorname{Int} A &= X - \mathbf{K}(X - \operatorname{Int} A) \\
&= X - \mathbf{K}(X - (X - \mathbf{K}(X - A))) \\
&= X - \mathbf{KK}(X - A) \\
&= X - \mathbf{K}(X - A) \\
&= \operatorname{Int} A
\end{aligned}$$

4d) $\operatorname{Int} X = X - \mathbf{K}(X - X) = X - \mathbf{K}\emptyset = X - \emptyset = X$. *Fin.*

It seems like the line of questioning that follows confused everyone. Here is the idea behind it. The above four theorems bear a strong resemblance to the closure operator axioms. This is no accident. As it turns out, we can make a theory of interior operators parallel to our theory of closure operators. Suppose (X, Int) is a pair consisting of a set X and a rule Int taking subsets of X to subsets of X such that Int satisfies the above four "theorems" (axioms, really, in this setting). Then we can *define* a closure operator $\mathbf{K}$ by setting $\mathbf{K}A = X - \operatorname{Int}(X - A)$. As it turns out, we can now prove the four closure "axioms" as theorems! As a result, writing down a pair $(X, \mathbf{K})$ is exactly the same as writing down the pair (X, Int); the two notions are on equal footing. No one wrote that they noticed this, but I can chalk that up to the (chronic!) vagueness of some of these homework questions. Oh well.