Math 101: Solution Set #3

First a note about the grading for this set: Each problem except for number 2 of §$N2.4$ was graded out of a possible 10 points – yielding a total of a possible 80 points. Since number 2 just asked you to do "as much as you can," I simply gave extra credit points for that based on how much you did.

A. If $X = \{a, b\}$ then there are four possible closure operators on X. Those are the discrete, the trivial and the two listed below:
For $A \subset X$ define **K**A as:

$$\mathbf{K}A = \begin{cases} \varnothing & \text{if } A = \varnothing \\ A \bigcup \{a\} & \text{if } A \neq \varnothing \end{cases}$$

The fourth closure operator is the same as the one listed above simply replacing $A \bigcup \{a\}$ with $A \bigcup \{b\}$. It is not to hard to check that these are indeed closure operators.

If we are dealing with a set of three elements then obviously the discrete and trivial operators are closure operators, but we will also have a whole host of operators defined like the one I defined above. However, there are a lot more possibilities because we can define the closure of specific sets to be certain things, just as long as we are sure that we satisfy all four of the axioms for closures. As we add more elements into our set the number of possible closure operators just grows and grows, and in fact when we get infinite sets (especially uncountable inifinite sets like $\mathbb{R}$) then we can have all sorts of different weird closures.

B. §N2.4

#1 **Part 1:** We want to show that $\bigcup_{i \in I} \mathbf{K}A_i \subset \mathbf{K} \bigcup_{i \in I} A_i$. To do this we pick an arbitrary element $j \in I$. We then note that by the definition of union, $A_j \subset \bigcup_{i \in I} A_i$. We then apply Theorem 1.9 to get that $\mathbf{K}A_j \subset \mathbf{K} \bigcup_{i \in I} A_i$. However, the choice of j was arbitrary, and hence this statement must be true for all $j \in A$ and hence for all A_j. Then by the definition of union we have that $\bigcup_{i \in I} \mathbf{K}A_i \subset \mathbf{K} \bigcup_{i \in I} A_i$

Part 2: We want to show that $\mathbf{K} \bigcap_{i \in I} A_i \subset \bigcap_{i \in I} \mathbf{K}A_i$. To do this we again pick an arbitrary element $j \in I$. By the definition of intersection we then have that $\bigcap_{i \in I} A_i \subset A_j$. Again, we apply Theorem 1.9 to get that $\mathbf{K} \bigcap_{i \in I} A_i \subset \mathbf{K}A_j$. However our choice of j was completely arbitrary and hence this must hole for all A_j. By the definition of intersection we therefore have that $\mathbf{K} \bigcap_{i \in I} A_i \subset \bigcap_{i \in I} \mathbf{K}A_i$

#2 This is a long problem, and I congratulate all of you who attempted, and for the most part succeeded in doing parts of it. For the sake of my own sanity and time I am not going to attempt to go through the entire proof here. However, if you have questions about any parts of it, or are curious about how you do any parts of it please come ask me some time. The main point of the excercise is to 1) show you that our definition and the "standard" definition of topology are indeed exactly the same, and 2) to give you some experience in doing formal manipulation of open and closed sets.

I will give a proof of part **(a)** here as an example of how you might approach this problem:
We are given a closure operator and told that $T = \{U \subset X | X - U is closed\}$. In order to prove that T is a topology we have to verify the three axioms:
(i): Is $X \in T$? Well X is in T iff $X - X$ is closed. But $X - X = \varnothing$, and $\varnothing$ is closed by axiom C1. Is $\varnothing \in T$? Well, it is iff $X - \varnothing$ is closed. However, $X - \varnothing = X$, and X is closed as we know from class. Hence $X, \varnothing \in T$.
(ii): Given $U, V \in T$ we want to show that $U \bigcap V \in T$. $U, V \in T$ implies that $X - U$ and $X - V$ are closed. Thus $(X - U) \bigcup (X - V)$ is closed since we know that the finite union of closed sets is closed. But we know that $(X - U) \bigcup (X - V) = X - (U \bigcap V)$ by Thm. 5.2(g) in our friend Mr. Wolf's book. This implies that $X - (U \bigcap V)$ is closed, and hence $U \bigcap V \in T$.
(iii): We assume that $U_i \in T \ \forall i \in I$. (Note that I is some arbitrary index set, so it is not necessarily finite or even countably infinite.) We want to show that $\bigcup_{i \in I} U_i \in T$. Well, $X - U_i$ is closed for all $i \in I$ by definition of T. Thus $\bigcap_{i \in I}(X - U_i)$ is closed since we know that the intersection of closed sets is closed. But we also know that $\bigcap_{i \in I}(X - U_i) = X - \bigcup_{i \in I} U_i$ by DeMorgan's Laws. Hence $X - \bigcup_{i \in I} U_i$ is closed, and hence $\bigcup_{i \in I} U_i \in T$

C. §W4.5

#6a We split the derivation of the formula for $\sum_{i=1}^{n} i$ into two cases:
n is even: If n is even then $\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n$ We can regroup this to get $\sum_{i=1}^{n} i = (1 + n) + (2 + (n - 1)) + \ldots + (\frac{n}{2} + (\frac{n}{2} + 1))$. We then have $\frac{n}{2}$ terms each of which is equal to $n + 1$ so then clearly $\sum_{i=1}^{n} i = \frac{n}{2}(n + 1)$.
n is odd: Then the sum can be regrouped as follows: $\sum_{i=1}^{n} i = 1 + 2 + 3 \ldots + n = (1 + n) + (2 + (n - 1)) + \ldots + (\frac{n-1}{2} + (\frac{n+1}{2} + 1)) + \frac{n+1}{2}$. We then clearly have $\frac{n-1}{2}$ terms of value $n + 1$ and one term of value $\frac{n+1}{2}$. Hence $\sum_{i=1}^{n} i = \frac{(n-1)}{2}(n + 1) + \frac{n+1}{2} = \frac{n^2-1+n+1}{2} = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$

#19 There were two ways in which to interpret this problem. One was to treat each greeting between two people as two distinct "Hellos," and the other was to consider it as one unique "Hello."
First way: We hypothesize that if there are n people and each has to say "hello" to everyone else once then each person will say "hello" to everyone in the group except themselves. Hence each person will say $n - 1$ "hellos," and hence there will be $n(n - 1)$ 'hellos" said. We prove this by induction:

Base Case: If $n = 1$ then there is one person, and clearly there is no one to say hello to, hence 0 hellos will be said. This agrees with our formula because $1(1 - 1) = 0$.

Inductive Hypothesis: We assume that our formula holds for all groups of people with n or less members.

Case of a group of $n + 1$ people: If there are $n + 1$ people then we can relate this to the Inductive Hypothesis by considering a group of n people, and then the $n + 1$th person walking in. In the group of n people, $n(n - 1)$ hellos will be said by our Inductive Hypothesis. When the new person walks in he or she will say hello to the n people already there, and each of the n people will say hello to the new person. Hence an additional $2n$ hellos will be said. So the total number of hellos will be $n(n-1)+2n = n^2-n+2n = n^2+n = n(n+1) = (n+1)((n+1)-1)$. Hence our hypothesis is proved.

Second way: We are considering each greeting between two people as a hello for both people. Hence the first person will have $n-1$ people to greet. The second, having already greeted the first, will then have $n-2$ people to great. And so on and so forth until the last person will have noone left to greet. We can see that this is the sum of the numbers from 0 up to $n-1$ and hence the total number of greetings will be $\frac{(n-1)n}{2}$. We prove this by induction:

Base Case: If $n=1$ then there is one person, and clearly there is no one to say hello to, hence 0 hellos will be said. This agrees with our formula because $\frac{(1-1)1}{2} = 0$.

Inductive Hypothesis: We assume that our formula holds for all groups of people with n or less members.

Case of a group of $n+1$ people: If there are $n+1$ people then we can relate this to the Inductive Hypothesis by considering a group of n people, and then the $n+1$th person walking in. In the group of n people, $\frac{n(n-1)}{2}$ hellos will be said by our Inductive Hypothesis. When the new person walks in he or she will greet to the n people already there. Hence an additional n hellos will be said. So the total number of hellos will be $\frac{n(n-1)}{2} + n = \frac{n^2-n+2n}{2} = \frac{n^2+n}{2} = \frac{n(n+1)}{2} = \frac{(n+1)((n+1)-1)}{2}$. Hence our hypothesis is proved.

§W5.3

#14a We are trying to show that $[P(0) \ \wedge \ \forall n \ (P(n) \rightarrow P(n+1))] \rightarrow \ \forall n \geq 0 \ P(n)$. Well, we note that $0 \in \mathbb{Z}$ and that $0+1 = 1$. Therefore since we know $P(0) \wedge \forall n \geq 0(P(n) \rightarrow P(n+1))$ then we know that $P(0) \rightarrow P(1)$. Hence we know P(1). Hence we know $[P(1) \wedge \ \forall n(P(n) \rightarrow P(n+1))]$. By the principle of mathematical induction we therefore know $\forall n P(n)$, and we already know P(0), hence we have that $\forall n \geq 0 \ P(n)$. (Another way in which to do this problem would be to define Q such that Q(n) = P($n-1$) and then to proceed from there.)

#17 I want to thank all of you who started their proofs by assuring me that the theorem was indeed false. I was getting a little worried that all horses might actually be the same color :-). Many of you tried to tell me all sorts of complicated things that went wrong, or to just say the proof was wrong because the theorem is incorrect, and gave a counter-example. While this shows the theorem is incorrect, it doesn't tell us why the proof is flawed. There are two main things we can point to to see why the proof fails. The first is to simply look at the first case after the base case of $n=1$. Clearly, as the proof says, the case of $n=1$ is trivial. So next consider the case $n=2$. Then their method fails Because then $A - \{c\} = k$ and $A - \{k\} = \{c\}$, so they share no elements and hence their method of proof fails. The second, and more major thing, that goes wrong is simply that the color of a horse, or the color of any number of horses has absolutely nothing to do with the n. Our statement is most certainly not a function of n in any way shape or form. (If you want to see another great "proof" about horses, this one proving that all horses have an infinite number of legs, check out `http://rhino.cee.odu.edu/sunny/ac/horse.html`)

D. §W3.2

#3 For #3 and #7 I have given possible solutions, there are many others that are equivalent to the ones listed.

(a) $\nexists m(m > 0 \wedge m < 1)$

(b) $\nexists n(n > m\ \forall m)$

(c) $\exists n(2n + 1 = m)$ (We let this proposition be equivalent to $\mathrm{O}(m)$)

(d) $\forall m, k\ (n = mk \rightarrow (m = 1 \vee m = n))$

(e) $\forall m\ (\mathrm{P}(m) \rightarrow (m = 2 \vee \mathrm{O}(m)))$

(f) $\forall m\ (\mathrm{P}(m) \rightarrow (\exists n(\mathrm{P}(n) \wedge n > m)))$

(g) $\forall x\ ((\forall n(x \neq n)) \rightarrow (\exists m\ (x < m < (x + 1))))$

(h) $\forall x, y\ (x \neq y \rightarrow \exists z((x < z < y) \vee (y < z < x)))$

#7 (a) $\neg W(x)$ (Define this to be $\mathrm{M}(x)$)

(b) $P(x, y) \wedge M(x)$

(c) $\exists z\ (P(x, z) \wedge P(z, y) \wedge W(x))$

(d) $\exists m, n\ (P(m, x) \wedge P(m, y) \wedge P(n, x) \wedge P(n, y) \wedge m \neq n \wedge x \neq y)$
(We define this property as $\mathrm{S}(x, y)$)

(e) $\nexists y\ (S(x, y))$

(f) $\exists m, n\ (S(m, n) \wedge P(m, x) \wedge P(n, y) \wedge \neg S(x, y) \wedge x \neq y)$
(I'm just making sure that x and y are not inbred. :-))

(g) $\forall m, n\ (P(m, x) \wedge S(m, n) \rightarrow W(n))$

(h) $\exists x\ (\exists y\ (S(x, y) \wedge M(y)) \wedge \nexists y\ (S(x, y) \wedge W(y)))$ (Note: For this one it is important to say that x does have brothers, but not sisters. It is not sufficient to simply say that x has no sisters.)