MATH 101 SOLUTION SET 6

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A. 2. *Suppose* $x_1, x_2 \in \mathbf{R}^n$ *and* $r_1, r_2 > 0$. *Use the triangle inequality to prove the following:*

a) *Show that* $d(x_1, x_2) \leq r_2 - r_1 \Rightarrow B(x_1; r_1) \subset B(x_2; r_2)$

Assume $d(x_1, x_2) \leq r_2 - r_1$. Suppose $y \in B(x_1; r_1)$. Then

$$\begin{aligned} d(y, x_2) &\leq d(y, x_1) + d(x_1, x_2) \text{ by the triangle inequality} \\ &< r_1 + d(x_1, x_2) \text{ by definition of } B(x_1; r_1) \\ &< r_1 + (r_2 - r_1) = r_2 \end{aligned}$$

Thus $y \in B(x_2; r_2)$ and the result follows.

b) *Show that* $d(x_1, x_2) \geq r_1 + r_2 \Rightarrow B(x_1; r_1) \cap B(x_2; r_2) = \emptyset$ Assume $d(x_1, x_2) \geq r_1 + r_2$. Suppose $y \in B(x_1; r_1)$. Then

$$\begin{aligned} d(y, x_2) &\geq d(x_1, x_2) - d(y, x_1) \text{ by the triangle inequality} \\ &> d(x_1, x_2) - r_1 \text{ by definition of } B(x_1; r_1) \\ &\geq (r_1 + r_2) - r_1 = r_2 \end{aligned}$$

Thus $y \notin B(x_2; r_2)$ and the result follows.

c) *Do you think the converses of these statements are true?*

They are true, and most of you drew the right pictures. A proof of this is a little tricky. Let me know if you want to see it.

3. *Given a nonempty* X, *show that you can define a metric by setting* $d(x, y) = 1$ *whenever* $x \neq y$ *in* X. *What closure operator does this induce?*

We can define

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

and it is trivial to show this is a metric. Suppose $A \subset X$. We claim $KA = A$ if K is the closure operator associated to this metric. It suffices to show $KA \subset A$. Suppose $x \in KA$. This means that for any $\epsilon > 0$ there's an $a \in A$ such that $d(x, a) < \epsilon$. In particular, there must be an $a \in A$ such that $d(x, a) < 1$. But by the construction of $d(x, y)$ this forces $d(x, a) = 0$; that is, $x = a$, implying that $x \in A$. Thus K is the discrete closure operator.

4. *Consider the set* $\mathbb{L} = \mathbf{Z}^+ \cup \{\infty\}$. *Show that you can define a metric on* $\mathbb{L}$ *by setting*

$$d(n,m) = \begin{cases} |1/n - 1/m| & \text{if } m,n \in \mathbf{Z}^+ \\ 1/n & \text{if } m = \infty \\ 1/m & \text{if } n = \infty \end{cases}$$

Let us make a convention that $1/\infty = 0$, so that the definition above reduces to $d(m,n) = |1/m - 1/n|$ for any $m, n \in \mathbb{L}$. Now look at the space $\{1/n \mid n \in \mathbf{Z}^+\} \cup \{0\}$, with the euclidean metric. By associating n to $1/n$, can you see that this space is really the "same" space as $\mathbb{L}$? The distances between points are the same; only the names of these points are different. We already know that the euclidean metric is a metric, so this observation takes care of the proof.

Prove that this metric induces the closure we have studied before on this space.

I saw a lot of references to the idea of "discrete distance," and the general line of reasoning that "since the distances $d(m,n)$ are discrete this is the discrete closure operator." I sense what is trying to be communicated, but these things don't make any sense. Topological spaces can be discrete. Distances can't. It's like talking about "finite numbers." The adjective just doesn't apply to that type of noun. Which leads me to another big problem, the mass confusion of "infinite set" with the symbol ∞. As I graded some of these papers you could see the steam rising from my forehead. $\{7, 15, \infty\}$ is a finite set which contains ∞. $\{4, 5, 6, \dots\}$ is an infinite set that does not contain ∞. ∞ is just a symbol.

Moving on. Let $A \subset X$. We wish to determine KA. First, we claim $KA \subset A \cup \{\infty\}$. If $x \notin A \cup \{\infty\}$, then for $a \in A$, then $d(x,a)$ is at least as big as the difference between x and its closest neighbor, namely $x+1$: $d(x,a) \geq d(x, x+1) = 1/x(x+1)$. (Was this what was meant by "discrete distance??") If $\epsilon = 1/x(x+1)$ then we can not find an $a \in A$ closer to x than ϵ, which means just that $x \notin KA$. Thus $KA \subset A \cup \{\infty\}$.

Will the reverse inclusion hold? Certainly $A \subset KA$. When will $\infty \in KA$? If $\infty \in A$ to begin with, then this is of course the case, so assume $\infty \notin A$. Then we claim $\infty \in KA$ exactly when A is an infinite set. If A is infinite, let $\epsilon > 0$. Since there is no bound to the elements of A there will be an $n \in A$ such that $n > 1/\epsilon$ (this is the archimedean property of the real numbers); then $d(n, \infty) = 1/n < \epsilon$. It follows that $\infty \in KA$, so that $KA = A \cup \{\infty\}$. On the other hand, if A is finite, then it has a greatest element, say n. Then for $a \in A$ we have $d(a, \infty) = 1/a \geq 1/n$ is bounded from below, so that $\infty \notin KA$. Thus $KA = A$ when A is finite. (In fact, this is always true in a metric space.)

B. 2. All of you found descent counterexamples.

4. All of you got this right.

5. *Define a function* d' *on pairs of points in* $\mathbf{R}^n$ *as follows:*

$$d'(x,y) = \sum_{i=1}^{n} |x_i - y_i| .$$

Check that d' *is a metric.* None of you had any problems with this part; the triangle inequality follows from the fact that $d(x,y) = |x - y|$ is a metric on $\mathbf{R}$ itself.

Now if $A \subset \mathbf{R}^n$, *define*

$$K'A = \{x \in \mathbf{R}^n \mid \text{ for all } \epsilon > 0 \text{ there exists } a \in A \text{ such that } d'(x,a) < \epsilon\} .$$

Show that $K'A = KA$ *for all* $A \subset \mathbf{R}^n$.

This was I think the trickiest problem on a math 101 set to date, in part because most of you haven't worked with the epsilon concept for very long, and it is almost universally a big hurdle. But still, I was sort of alarmed at how many of you mixed up the order of the quantifiers in the above definition. When it gets too confusing, just think: "$x \in KA$ when the points in A come closer to x (with respect to d') than any distance you can name."

I'll prove something general about comparing two metrics on the same space:

Suppose d and d' are two metrics on a set X. Suppose further that there are constants C and C' such that for all $x, y \in X$ we have

$$d'(x,y) \leq Cd(x,y) \text{ and } d(x,y) \leq C'd(x,y).$$

Then the closure operators coming from d and d' are equivalent.

Here's the proof. Let K and K' be the closure operators coming from d and d'. Let $A \subset X$. We shall prove $KA \subset K'A$. Let $x \in KA$. To prove that $x \in K'A$ as well, we need to show that the points of A come closer (with respect to d') than any distance ϵ you can name. Suppose such an ϵ is given. Now since $x \in KA$, there should be a point a of A closer (with respect to d) to x than ϵ/C. Then $d'(x,a) \leq Cd(x,y) < C(\epsilon/C) = \epsilon$. Great, we found an $a \in A$ which was closer (with respect to d') than ϵ. So $x \in K'A$ as required. By symmetry this also shows that $KA \subset K'A$, so in fact $KA = K'A$.

Now for the problem at hand, if we can show that there's a C and a C' for which $d(x,y) \leq C'd'(x,y)$ and $d'(x,y) \leq Cd(x,y)$, then we can apply the theorem and we'd be done. We claim we can take $C' = 1$ and $C = n$. For the first claim, just note that

$$d'(x,y)^2 = \left(\sum_{i=1}^{n} |x_i - y_i|\right)^2 \geq \sum_{i=1}^{n}(x_i - y_i)^2 = d(x,y)^2,$$

and take the square root of both sides. For the second claim, note that for each i, $|x_i - y_i|^2 \leq \sum_{i=1}^{n}(x_i - y_i)^2 = d(x,y)^2$, so that $|x_i - y_i| \leq d(x,y)$. Sum this inequality over all i to get

$$d'(x,y) = \sum_{i=1}^{n} |x_i - y_i| \leq nd(x,y).$$

C. 1. *Let a and b be any positive numbers. By working backwards, find proofs of the following three inequalities...*

None of you had much problem doing this. I'll remark, however, that the broadest generalization of this sort of expression is the pth-power mean, defined for nonnegative numbers $x_1, \ldots, x_n$ and any p such that $p \neq 0$:

$$M_p(x_1, \ldots, x_n) = \left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{1/p}$$

This is undefined for $p = 0$ but you can use L'Hôpital's rule to show that this expression tends to the geometric mean $(x_1 x_2 \ldots x_n)^{1/n}$ as p goes to 0. When $p = -1, 1, 2$ then M_p reduces to the harmonic mean, arithmetic mean, and root-mean-square respectively. A little calculation can be done to show that $(d/dp)M_p$ is always positive and so M_p is strictly increasing, except when all the x_i are equal, in which case M_p always reduces to their common value.

A nice group project might run as follows. Fill in the details of the above, and show furthmore that for $p > 0$ you can define a metric d_p on $\mathbf{R}^n$ as

$$d_p(x,y) = M_p(|x_1 - y_1|, \ldots, |x_n - y_n|).$$

Then d_p will actually be a metric. Note that d_2 is proportional to the euclidean metric and d_1 is proportional to the metric d' we considered two problems back. Show that all the d_p are equivalent in the sense of two problems back. What happens when $p \to \infty$?.

The Wolf problem was done correctly by almost everyone, although the order-of-quantifiers issue it addresses still needs attention.