Math 101!
Hour and a Half Exam

Closed book, in class, on November 29, 2001.

Each of the first 8 problems is worth 12 points. They are divided into 4 parts, and you can earn one extra point for good style in each part. The total number of points available is therefore 100.

You can do Problem 9 as a substitute for one of the first 8 if you indicate this on the scoring grid of the examination book cover by circling 9 and crossing off the number of the problem it is replacing.

Part A: The Least Upper Bound Principle.

1. Let A denote a nonempty subset of the real line. State a necessary and sufficient condition on A for the existence of a real number that is a least upper bound for A. Can there be more than one least upper bound for A? Prove your assertion.

2. Show that if a real number x is a least upper bound for A, then x belongs to $\mathbf{K}A$, where $\mathbf{K}A$ denotes the standard Euclidean closure of A.

Part B: Are "Open Balls" Really Open?

3. Let (X, d) be a metric space. Fix x in X and $r > 0$. As usual, the open ball of radius r centered at z is denoted $B(z; r)$. Prove that if x belongs to $B(z; r)$, then there exists an $\varepsilon > 0$ such that $B(x; \varepsilon) \subset B(z; r)$. A sketch may help you get started.

4. State the contrapositive of the implication you just proved. It may be helpful to formulate it in terms of the set $A = X - B(z; r)$. Then explain how you can conclude that $B(z; r)$ really is open as we have defined the term "open" using closure operators.

Part C: Connected Components.

5. Fix an element x of a topological space (X, $\mathbf{K}$). Show that there exists a subset D of X that is the largest connected set containing x. You should explain what you mean by largest, describe a construction, and quote carefully but without proof the theorem you need to justify your construction.

6. Show that if D is the largest connected subset of X containing a fixed x, then D is closed. To do this, you will need to state and prove a lemma about closures of connected sets.

Part D: Power Sets by Induction.

7. Let $A_n = \{ 1, 2, 3, \ldots, n \}$. It follows by induction, I assert, that since the power set $\wp(A_n)$ is countable, then the power set of the positive integers, $\wp(\mathbf{Z}^+)$, is also countable. Here is my proof:

I. Induction on n shows that $\wp(A_n)$ has $2^n - 1$ elements.
II. A countable union of finite sets is countable.
III. Therefore $\wp(\mathbf{Z}^+)$ is countable.

Critique my argument by answering the following questions: Are the premises (I) and (II) true statements on their own? For each, fix whatever is not right and give a brief proof of your corrected version if you make any changes. Does (III) follow from the premises as you have fixed them? Why or why not?

8. Is $\wp(\mathbf{Z}^+)$ countable or not? Give a convincing proof.

Part E: Wild Card that Can Substitute for any One Problem Above.

9. Define carefully what it means to say that a function f is continuous. State your favorite theorem or exercise result concerning continuous functions, then prove it. Explain in your own words what makes this result significant, interesting, or useful.