Chapter 2

Smallest, Infinite, and Closed Sets

The Brouwer Fixed Point Theorem talks about how certain kinds of transformations of certain kinds of sets must leave at least one point where it started. To see what ingredients we will need to prove such a theorem, consider some similar cases in which such a result would not hold. This will lead us to four conditions we must assume about our set and its transformation to obtain the theorem we seek: connectedness, completeness, continuity, and compactness. Although it will take us chapters to formalize precise definitions, we can see roughly what these necessary conditions must be like by going back to our motivation about taking a sheet of paper taken from a pad, crumpled and put back down on the pad:

For example, if our paper were allowed to have a hole in it, the fixed point we seek might be missing. When rotated a quarter turn, a square piece of paper punctured at its center contains no fixed point. Intuitively, saying a set is "complete" will mean that any holes, even ones we cannot name, are all filled in.

If we allowed ourselves to imagine paper extending indefinitely like a plane in all directions, then just sliding it along would leave no fixed points. Requiring the sets we work with to be "compact" will prevent them from spilling outwards without end like this.

If we started with a set consisting of two pieces of paper on two pads, then we could just switch them and not leave anything fixed. The notion of "connectedness" will help us capture the idea that the sets we want should be all of one piece.

Finally, if we allow ourselves to rip the paper, then again all bets are off concerning fixed points, since we could make holes or separate pieces even if none were there at the onset. Requiring our transformation to be "continuous" is how we will say that it can bend, stretch, fold, shrink, and do lots of other things but it cannot tear.

Formulating any of these four concepts much more precisely will require a notion of closeness. In other words, we will need to work in a topological space $(X, \mathbf{K})$ as defined in the previous chapter. Even before getting to completeness, we can already say in this situation what it means for a set to have nothing obvious missing.

Definition 2.1 *Suppose* $(X, \mathbf{K})$ *is a topological space and* $A \subset X$. *We say that* A *is* **closed** *in* $(X, \mathbf{K})$ *if* $\mathbf{K}A = A$.

For the Euclidean closure operator $\mathbf{K}_e$ on the line we have imagined as motivation, this usage is consistent with the fact that intervals like $[a, b]$ that contain their endpoints are usually called closed intervals. For the Euclidean closure operator in the plane, the punctured square imagined above would certainly not be closed since its closure would include the missing point. This example should make us expect that, whatever other hypotheses go into proving the existence of fixed points, we will want to be dealing with sets that are closed.

Notice that it only makes sense to say that a set is closed with respect to a given closure operator. In other words, a particular set of points A could be left fixed by closure operator $\mathbf{K}_1$ and hence be closed in $(X, \mathbf{K}_1)$ but not be closed in $(X, \mathbf{K}_2)$, a different topological space formed out of the same X by equipping it with a different closure operator. For example, the only closed subsets of X topologized by the trivial closure operator are the empty set $\emptyset$ (by Axiom 4) and the whole space X (by our first fundamental result). On the other hand, every subset of X is closed under the discrete closure operator.

In this chapter, we will develop another characterization of the closure of a set A as the smallest closed set containing A. So that we can take unions and intersections of lots of closed sets, we will have to think about families of sets and how to index them. This involves getting serious about properties of the counting numbers. In particular, we will distinguish among: sets whose members are easy to imagine counting up because they are finite; infinite sets we can count in principle even if we never finish; and sets that are so big it does not even make sense to talk about counting their elements. In the end, we will see that our axioms and definitions force us to admit closed sets on the line under the Euclidean Closure that are much stranger looking than intervals containing their endpoints.

2.1 Smallest sets

To begin making sense out of the intuition that the closure of A should be the smallest closed set containing A, consider the following definition and lemma.

Definition 2.2 *Let F denote a set of sets and let P denote a predicate on F, meaning that for each set $A \in F$ either $P(A) = T$ or $P(A) = F$ but not both. We say that B is the smallest set in F satisfying P if $P(B) = T$ and $P(A) = T$ implies $B \subset A$ for all $A \in F$.*

A similar definition for the largest set results by reversing the set inclusion. The family F might typically be the power set 2^X of a set X consisting of all the subsets of X. For example, if $F = 2^{\mathbb{Z}}$ and $P(A) = T$ if and only if $7 \in A$, then $\{7\}$ is the smallest subset of the integers $\mathbb{Z}$ satisfying P and $\mathbb{Z}$ is the largest. On the other hand, for the predicate $P(A) = T$ if and only if $7 \notin A$ on the same F, there is no smallest set satisfying P.

The question of the existence of sets defined in this way deserves some care. Logicians might object that we have given an "impredicative" definition because of the way we are attempting to determine an object in terms of a set that contains the object being defined. Here the object we wish to characterize is the smallest set with a given property, but we do so using a set to which it is supposed to belong, namely the set of all sets with that property. As mentioned in the previous chapter, moves like this are what generate Russell's Paradox as well as other potential conundrums and contradictions.

Despite the need for caution to avoid a vicious circle, the form of definition given above is quite useful in practice, as we shall see in several different contexts. For example, fix an $A^* \subset X$ in a topological space $(X, \mathbf{K})$. We claim that, if there does exist a smallest closed set containing this A^*, it must equal $\mathbf{K}(A^*)$. To see this, set $F = \{A \in 2^X \mid A^* \subset A\}$, the family of all supersets of A^* and consider the predicate P on this F with $P(A) = T$ if and only if $\mathbf{K}A = A$, or, in other words, the predicate that is true if and only if A is closed. In this situation, suppose that the smallest set in F satisfying P exists and call it B. Then, by definition of what it means to be the smallest set, we have $B \subset \mathbf{K}(A^*)$ because, $\mathbf{K}(A^*)$ is a closed set containing A^*. On the other hand, we also have $\mathbf{K}(A^*) \subset B$ by Theorem 1.9 because $A^* \subset B$ implies $\mathbf{K}(A^*) \subset \mathbf{K}B = B$. We conclude that $\mathbf{K}(A^*) = B$ as claimed.

The point is that the definition above of the smallest set with a given property can be quite convenient way of proving results – if we know whether such a set exists. A general approach to the existence question might lead us to Zorn's Lemma, which is called a lemma but is really an axiom since it is not a logical consequence of the usual axioms of mathematics. For our purpose, namely the characterization of closure as the smallest closed superset, it is more straightforward and useful to reformulate our definition in a way that seems less suspicious.

Lemma 2.3 (Smallest Set Lemma) *Let F denote a set of sets and let P denote a predicate on F. If the smallest set in F satisfying P exists, then it is unique and equals the intersection of all $A \in F$ satisfying $P(A) = T$. On the* *other hand, if the intersection of all $A \in F$ satisfying $P(A) = T$ is a set D such that $P(D) = T$, then D is the smallest set in F satisfying P.*

In other words, if such a smallest set exists, we claim that it can be written as

$$\{x \in X \mid \exists A \in F \text{ with x} \in \text{A and P(A)} = \text{T}\} = \bigcap_{\text{A} \in \text{G}} \text{A}$$

where we set $X = \bigcup_{A \in F} A$ is the union of all the sets in the family F and $G = \{A \in F \mid P(A) = T\}$, the "truth set" of P.

Proof. Suppose that B, the smallest set of F satisfying P, exists and let D denote the intersection of all $A \in F$ satisfying $P(A) = T$. Because $P(B) = T$, we see that B is one of the sets intersected to form D and so $D \subset B$. Because B is a subset of every set used in forming this intersection, we also have $B \subset D$. Hence $B = D$ as claimed. The uniqueness of B follows from the definition of what we mean by a smallest set since, if there were two, each would have to be a subset of the other. The last statement follows from the definition of the smallest set and the definition of intersection.

□

Recall that, given any set A in a topological space $(X, \mathbf{K})$, we would like to characterize $\mathbf{K}A$ as the smallest closed subset of X containing A. Does the smallest closed set B containing A exist? Closed sets containing A certainly do exist, including $\mathbf{K}A$ and X itself. Moreover, the intersection of all of them must still contain A. The question now is whether or not the intersection of lots of closed sets must be closed. We therefore turn to studying what does happen if we take intersections and unions of families of closed sets. When considering a set of sets F, it is convenient to assume that it is indexed by some set I, meaning that $A \in F$ if and only if $A = A_i$ for some $i \in I$. In terms of this index set, we write

$$\bigcap_{A \in F} A = \bigcap_{i \in I} A_i = \{x \mid (\forall i \in I)\ x \in A_i\}$$

$$\bigcup_{A \in F} A = \bigcup_{i \in I} A_i = \{x \mid (\exists i \in I)\ x \in A_i\}$$

to denote the intersection and union of all members of F.

2.2 The finite case

We call a nonempty set X finite if it has an index set of the form $I = \{1, 2, 3, \ldots, n\}$ for some $n \in \mathbb{Z}^+$, meaning that we can write

$$X = \bigcup_{i \in \{1,2,\ldots,n\}} \{x_i\}$$

with $x_i \in X$ for all $i \in \{1, 2, \ldots, n\}$. In particular, suppose that we have a finite family $F = \{A_1, A_2, \ldots, A_n\}$ of closed sets within a given topological space. What can we say about the intersection

$$\cap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

and union

$$\cup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

of the sets in this family?

From Axiom 2, we know that

$$\mathbf{K}(A \cup B) = \mathbf{K}A \cup \mathbf{K}B$$

so clearly, if A and B are closed, so is their union. Then, in Exercise 1 of the last section, you attempted to generalize the result to closures of unions of more than two sets, e.g.,

$$\mathbf{K}(A_1 \cup A_2 \cup A_3) = \mathbf{K}A_1 \cup \mathbf{K}A_2 \cup \mathbf{K}A_3$$

$$\mathbf{K}(A_1 \cup A_2 \cup A_3 \cup A_4) = \mathbf{K}A_1 \cup \mathbf{K}A_2 \cup \mathbf{K}A_3 \cup \mathbf{K}A_4$$

and so forth. The trick, of course, is to reduce each new case to one you have already handled. For three closed sets, let $A = A_1 \cup A_2$. This is the union of two closed sets, hence it is closed as we have seen. Then let $B = A_3$ and apply the same argument again to conclude that the union of the two closed sets A and B is closed. Similarly, we can group together the union of four closed sets as the union of three, which we just showed must be closed, with the fourth. This exhibits the total union as the union of two closed sets. Therefore it must be closed. It seems clear that we can repeat this process any number of times before halting; but how do we make the "and so forth" stage of the proof more precise? How can we be sure that, for each given $n \in \mathbb{Z}^+$, the union of n closed sets really is closed?

A similar question arises when we consider intersections of closed sets. If A and B satisfy $\mathbf{K}A = A$ and $\mathbf{K}B = B$, then by Axiom 1 and 1.10, we have

$$A \cap B \subset \mathbf{K}(A \cap B) \subset \mathbf{K}A \cap \mathbf{K}B = A \cap B$$

and so $A \cap B$ equals $\mathbf{K}(A \cap B)$, proving that the closure of the intersection of two closed sets is closed. Now given three closed sets, we can again group them as above to see that the intersection of the first two is closed, and then that the intersection of this set with the third must also be closed. Once we know this, we can use it to treat the case $n = 4$, and so on up the line. We could write out the argument for small values of n, but the argument strongly suggests that, for each given $n \in \mathbb{Z}^+$, the intersection of n closed sets really is closed. How can we be sure of really proving this? After all, we are not dealing with a finite number of assertions. Let P denote the predicate on $\mathbb{Z}^+$ defined by setting $P(n) = T$ if and only if the intersection of n closed sets is closed. How can we more formally prove our belief that $P(n) = T$ for all $n \in \mathbb{Z}^+$?

The answer is to use the **Principle of Mathematical Induction**. The principle is actually one of the famed **Peano Axioms** that state the most important properties we expect the set of natural numbers $\{1, 2, 3, \cdots\}$ to possess. The form of the induction principle we will use is stated as:

Definition 2.4 *(The Principle of Mathematical Induction) If* $P(1)$ *and* $(\forall n \in \mathbb{Z}^+)\ P(n) \Longrightarrow P(n+1)$*, then* $(\forall n \in \mathbb{Z}^+)\ P(n)$.

This axiom seems reasonable enough; if you like, you can imagine a row of dominoes standing up; when you topple the first one onto the second, then each succeeding one will fall, no matter how many dominoes are in the line.

Let us now put our new bit of mathematical machinery to work on proving that the union of finitely many closed sets is closed. We need to verify two things: (1) if a given set is closed, then that set is closed; and (2) if the union of n closed sets is closed, then the union of $n+1$ sets is closed. Statement (1) is a tautology and therefore holds. To see that statement (2) is true, call the union of the first n closed sets A. By assumption, A is closed. By the second closure operator axiom, the union of the $n+1$-th set and A must also be closed. Since the conditions of the principle of induction are satisfied, the statement holds true for any finite number of sets.

Here is another way to look at the Principle of Mathematical Induction. Call a $A \subset \mathbb{Z}$ **progressive** if $1 \in A$ and $n \in A$ implies $n+1 \in A$. In these terms, the Principle of Mathematical Induction is equivalent to the assertion that $\mathbb{Z}^+$ is the smallest progressive subset of of $\mathbb{Z}$: given the version that talks about sets, we can deduce the one that talks about predicates by setting A equal to the truth set of P; conversely, the set version follows from the one about predicates if we apply it to to the predicate P that is true of n if and only if $n \in A$. (For yet another equivalent formulation, look up the Well Ordering Principle.)

The Principle of Mathematical Induction is remarkable because it allows us to prove infinitely many statements $P(1), P(2), P(3), \ldots$ all at once. Note, however, that each of these is only a statement about a finite value of n. In particular, we have proven nothing so far about what happens when we take infinite unions or intersections of closed sets even though we have proven infinitely many cases about finite unions and intersections.

2.3 The Countable Case

We call a nonempty set X countable if it has an index set of the form $I = \mathbb{Z}^+ = \{1, 2, 3, \dots\}$ for some, meaning that we can write

$$X = \bigcup_{i \in \mathbb{Z}^+} \{x_i\}$$

with $x_i \in X$ for all $i \in \mathbb{Z}^+$. For example, the set of all integers $\mathbb{Z}$ is countable because

$$\mathbb{Z} = \bigcup_{i \in \mathbb{Z}^+} \{x_i\}$$

with $x_i = (i-1)/2$ for i odd and $x_i = -i/2$ for i even. (By the way, if you are wondering how to define $\mathbb{Z}$ once you have the Peano Axioms for $\mathbb{Z}^+$, a good way to characterize the integers is as the smallest "ring" containing $\mathbb{Z}^+$.)

The set of all ordered pairs (m, n) of positive integers, denoted $\mathbb{Z}^+ \times \mathbb{Z}^+$ is also countable, though this is perhaps not as obvious. As an exercise, you can imagine making a list of all these pairs by first placing them in an array, then moving up and down the diagonals on which $m + n$ is constant starting with $(1, 1)$.

Suppose that we have a countable family $F = \{A_1, A_2, A_3, \dots\}$ of closed sets within a given topological space. What can we say about the intersection

$$\cap_{i=1}^{\infty}\{A_i\} = \{A_1\} \cap \{A_2\} \cap \{A_3\} \dots$$

and union

$$\cup_{i=1}^{\infty}\{A_i\} = \{A_1\} \cup \{A_2\} \cup \{A_3\} \dots$$

of the sets in this family?

The first thing to notice is that the Principle of Mathematical Induction does not help answer such questions. Expressed in terms of the predicate $P(n)$ that is true if and only if the union of n closed sets is always closed, the assertion that the union of a countable infinity of closed sets is always closed is not of the form $P(n)$ for some $n \in \mathbb{Z}^+$. You can think of it as $P(\infty)$ if you like, but the statement is false in general as the following example shows.

Example 2.5 Consider the topological space $(\mathbb{L}, \mathbf{K})$ defined in Exercise 2 with $\mathbb{L} = \mathbb{Z}^+ \cup \{\infty\}$ and

$$\mathbf{K}A = \begin{cases} A & \text{if } A \text{ is finite,} \\ A \cup \{\infty\} & \text{if } A \text{ is infinite.} \end{cases}$$

In this space, every singleton $\{i\} \subset \mathbb{Z}^+$ is closed, and so $\mathbb{Z}^+$ is the countable union of closed sets. Yet $\mathbb{Z}^+$ is not closed as a subset of $\mathbb{L}$ since, in this topology, $\mathbf{K}(\mathbb{Z}^+) = \mathbb{L}$.

You can also think of examples using the Euclidean Closure Operator $\mathbf{K}_e$ that we have imagined (but not yet defined) to convince yourself that the infinite union of closed sets need not be closed. What about intersections? It turns out that intersections of arbitrarily many closed sets must be closed, as we will see in the next section. We will also see there that considering the case of countable intersections is not enough to justify the existence of a smallest closed set containing a given set.

2.4 The uncountable case

Recall that we wanted to justify the existence of a smallest closed set containing A by showing that the intersection of all the closed sets contain A is closed. We claim that there may be more than countably many closed sets containing A to intersect.

Example 2.6 Consider the topological space $(\mathbb{Z}^+, \mathbf{K})$ where $\mathbf{K}$ denotes the discrete closure operator that makes every $A \subset \mathbb{Z}^+$ closed. In other words, the set of all closed subsets of this space is $2^{\mathbb{Z}^+}$, the power set of the positive integers. We claim this power set is not countable. For suppose on the countrary that we could write

$$2^{\mathbb{Z}^+} = \{A_1\} \cup \{A_2\} \cup \{A_3\} \cup \dots$$

for some countable collection of sets $A_i \in 2^{\mathbb{Z}^+}$ indexed by $\mathbb{Z}^+$. Then we could form the set

$$B = \{i \in \mathbb{Z}^+ \mid i \notin A_i\}.$$

Because this B is a member of $2^{\mathbb{Z}^+}$, there would have to exist an n in the index set $\mathbb{Z}^+$ such that $B = A_n$. Now we ask whether or not this n belongs to B. Either answer leads to a contradiction, so it must be impossible to express $2^{\mathbb{Z}^+}$ as a countable union of sets.

This shows directly that the set of closed subsets of $(\mathbb{Z}^+, \mathbf{K})$ containing the empty set $\emptyset$ is not countable. The argument can be modified to show that the set of closed sets containing $\{1\}$, for example, is not countable, either. This is the family we would have to intersect if computing $\mathbf{K}\{1\}$ by the (silly in this case) method we have been studying.

It follows that we must consider index sets I more general than just $\mathbb{Z}^+$. What we can prove is that if $\{A_i \mid i \in I\}$ is any nonempty family of closed sets, then $\bigcap_{i\in I} A_i$ is closed. To see this, suppose j is a fixed element of I. Then $\bigcap_{i\in I} A_i \subset A_j$, by definition of intersection. By Theorem 1.9, it follows that $\mathbf{K}\bigcap_{i\in I} A_i \subset \mathbf{K}A_j = A_j$. Now this last inclusion is true for *every* $j \in I$, which means exactly that $\mathbf{K}\bigcap_{i\in I} A_i \subset \bigcap_{i\in I} A_i$. By Axiom C1, $\bigcap_{i\in I} A_i$ is closed.

As a corollary, we can finally assert that the intersection of all the closed sets containing a given set A is closed, and so the smallest closed subset of X containing A exists and equals $\mathbf{K}A$, the the closure of A. The following omnibus theorem summarizes the results we have found in this chapter so far.

Theorem 2.7 (Closed Set Theorem) *Let* $(X, \mathbf{K})$ *be any topological space. Then in this space,*

(a) X *and* $\emptyset$ *are closed.*

(b) If $A_1, \ldots, A_n$ *are closed, then* $A_1 \cup \cdots \cup A_n$ *is closed.*

(c) If $\{A_i \mid i \in I\}$ *is a nonempty family of closed sets, then* $\bigcap_{i \in I} A_i$ *is closed.*

(d) The closure $\mathbf{K}(A)$ *of any* $A \subset X$ *is the smallest closed subset of* X *containing* A.

The last item in this theorem says that, not only does a closure operator determine what we consider closed sets, but knowing all the closed sets also determines the closure operator. This should lead you to wonder whether, if you just declared some subsets of a given X to be closed, can you define a closure operator that does make those and only those sets closed? A good answer to this question not only uses the results in this chapter, but also will help you appreciate the standard way of defining a topological space that does not mention closure operators. (See the exercises below.)

Exercises

1. Let $(X, \mathbf{K})$ be a topological space and let $\{A_i \mid i \in I\}$ be a family of (not necessarily closed) subsets of X. Show that $\bigcup_{i \in I} \mathbf{K}A_i \subset \mathbf{K} \bigcup_{i \in I} A_i$ and $\mathbf{K} \bigcap_{i \in I} A_i \subset \bigcap_{i \in I} \mathbf{K}A_i$. (Note that this last statement is a generalization of Theorem 1.10.)

2. (harder)

   Let X be a set. A **topology** on X, denoted by T, is a set of subsets of X (i.e. $U \in T \Longrightarrow U \subset X$) such that:

   (i) $\phi, X \in T$.

   (ii) If $U, V \in T$ then $U \cap V \in T$.

   (iii) If $\{U_i \mid i \in I\}$ is a subset of T (i.e. $U_i \in T$ for each $i \in I$), then $\bigcup_{i \in I} U_i \in T$.

   The standard definition of a topological space is a set X along with a topology T on X. The elements of T are called **open sets**.

   This definition is equivalent to our nonstandard definition in the following sense.

(a) Suppose $\mathbf{K}$ is a closure operator on X. Let

$$T = \{U \subset X \mid X - U \text{ is closed}\}.$$

Show that T is a topology on X. *Hint:* This follows quickly from certain facts we have already proved.

(b) Suppose T is a topology on X. If $A \subset X$, define

$$\mathbf{K}A = X - \bigcup_{\{V \in T \mid V \cap A = \emptyset\}} V.$$

Show that $\mathbf{K}$ is a closure operator on X. *Hint:* The proof of C1 does not require any part of the definition of a topology; neither does the proof that $\mathbf{K}A \cup \mathbf{K}B \subset \mathbf{K}(A \cup B)$. To prove the other part of C2, use (ii) (after a certain amount of sorting through the definitions, as usual). To prove C3, use (iii). C4 is easier and uses (i).

(c) Show that if you start with a closure operator, derive a topology from it as in (a), and then derive a closure operator from this topology as in (b), then you get back the closure operator you started with. *Hint:* this is equivalent to an earlier exercise.

(d) Show that if you start with a topology, derive a closure operator from it as in (b), and then derive a topology from this closure operator as in (a), then you get back the topology you started with. *Hint:* use (iii).

2.5 The Cantor middle-thirds set*

The **Cantor set** is a classic example of a closed set constructed using infinite intersections on the real line with the Euclidean closure operator $\mathbf{K}_e$. To obtain this set, start with $[0,1]$. Remove the middle third of the interval, $(1/3, 2/3)$, to obtain the set $[0,1/3] \cup [2/3,1]$. Next remove the middle third of each of the remaining two intervals (i.e. (1/9,2/9) and (7/9,8/9)) to obtain the set $[0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$. Then remove the middle third of each of the remaing four intervals, and so on. If we "continue this process forever," the Cantor set is what remains.

More precisely, define

$$\begin{aligned} A_0 &= [0,1], \\ A_{n+1} &= \frac{1}{3}A_n \cup \left(\frac{2}{3} + \frac{1}{3}A_n\right). \end{aligned}$$

The second line is shorthand indicating that A_{n+1} is obtained by scaling A_n down to one-third size and putting one copy in $[0,1/3]$ and another in $[2/3,1]$. The two

Figure 2.1: The first three stages in the construction of the Cantor set.

lines together are an example of a **recursive definition**. By induction, A_n is defined for all $n \in \mathbb{Z}^+$. (For recursive definitions of addition and multiplication, see §C.3.) The first few values of A_n are:

$$\begin{aligned} A_1 =& [0, 1/3] \cup [2/3, 1] \\ A_2 =& [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1] \\ A_3 =& [0, 1/27] \cup [2/27, 1/9] \cup [2/9, 7/27] \cup [8/27, 1/3] \\ & \cup [2/3, 19/27] \cup [20/27, 7/9] \cup [8/9, 25/27] \cup [26/27, 1] \end{aligned}$$

(See Figure 2.1.)

Define the Cantor set to be

$$C = \bigcap_{n \in \mathbb{Z}^+} A_n.$$

We can use the Closed Set Theorem to show that C is closed. By induction, each A_n is the union of finitely many closed intervals, and hence closed, by part (b) of the Closed Set Theorem. By (c), C is closed. Not only is it nonempty, it is uncountable (see Exercise below). In this sense, the Cantor set is big. On the other hand, there is a sense in which the Cantor set is what remains after we have removed almost everything from the interval. More precisely, let's calculate the sum S_n of the lengths of all the middle thirds removed through stage n of our construction. At the first stage with $n = 1$, we take out an interval of length $\frac{1}{3}$. At the next stage, we remove two intervals of length $\frac{1}{3}$, then four intervals of length $\frac{1}{27}$, etc. Thus, we find

$$
\begin{aligned}
S_1 &= \frac{1}{3} \\
S_2 &= \frac{1}{3} + 2(\frac{1}{3}) \\
S_3 &= \frac{1}{3} + 2(\frac{1}{3}) + 4(\frac{1}{27}) \\
&\cdots \\
S_n &= \frac{1}{3} + 2(\frac{1}{3}) + 4(\frac{1}{27}) + \cdots + 2^{n-1}(\frac{1}{3^n})
\end{aligned}
$$

This sum is a finite geometric series as described by the following result if we set $a = \frac{1}{3}$ and $r = \frac{2}{3}$.

Lemma 2.8 (Geometric Series Lemma) *If a and r are real numbers with $r \neq 1$, and if $n \in \mathbb{Z}^+$, then*

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a - ar^n}{1 - r}.$$

Consequently, if $0 < r < 1$ and $a > 0$, then

$$a + ar + ar^2 + \cdots + ar^{n-1} < \frac{a}{1 - r}$$

and the difference between the right and left sides of this inequality can be made as small as you like by taking n sufficiently large.

The proof is a straightforward exercise using induction. Applying the Lemma with $a = \frac{1}{3}$ and $r = \frac{2}{3}$, we see that S_n grows as close as you like to one as n gets bigger and bigger. In other words, the length of the subset of the unit interval left for the Cantor set to sit in grows smaller and smaller at each stage. It seems reasonable to conclude that the Cantor set has length zero.

Exercises

1. Prove the Geometric Series Lemma using induction.

2. Prove that $1/4 \in C$. *Hint:* Show that if $1/4 \notin A_n$, then $3/4 \notin A_{n-1}$. On the other hand, show that $3/4 \notin A_n \implies 1/4 \notin A_{n-1}$. Use the Well-Ordering Principle to deduce a contradiction if $1/4 \notin C$.

3. Find a rule for determiniming when a point in $[0, 1]$ is an element of the Cantor set. *Hint:* use base 3 notation.

4. Show that the Cantor set is uncountable. One way to do this is to notice that each point in the Cantor set is the intersection of nested closed intervals. To every such point x there corresponds an element A of $2^{\mathbb{Z}^+}$ and vice versa determined by requiring $n \in A$ if and only if x belongs to the left hand of the two intervals that remain when their middle third is removed at stage n.