Math 101
Selected Solutions for Problem Set 3

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Problem W 5.3.11

Assuming that $0 \notin \mathbb{N}$, then

$$\bigcup_{n \in \mathbb{N}} A_n = (0,1).$$

To prove this note first that $A_n \subset (0,1)$ for all n, so that we have $\bigcup_{n \in \mathbb{N}} A_n \subseteq (0,1)$. For any $x \in (0,1)$, we know by the Archimidean property of the real numbers that $2^{-n} < x$ for some n in $\mathbb{N}$. Pick such n to be minimal. Then $x \in A_n$. So $\bigcup_{n \in \mathbb{N}} A_n \supseteq (0,1)$ also and we know the sets are equal as claimed. Finally, we see that

$$[0,1] - \bigcup_{n \in \mathbb{N}} A_n = \{0,1\}.$$

Many people tried to use limits in this problem and various arguments about how $\lim_{n \to \infty} 2^{-n} = 0$, but it never quite gets there. Since we haven't even considered the notion of a limit in the class, we certainly can't use limits to show anything at this point. It is good that this problem brings limits to mind as they are at least relevant in spirit, but this problem is simply a matter of demonstrating the equality of two sets—which pretty much never has anything to do with limits.

Problem W 8.1.1

It is standard to require the existence of a multiplicative identity in the definition of a ring. With this requirement, we can simply observe that $-1x = -x$. While this may seems obvious, it is not and requires proof:

$$x + (-1)x = 1x + (-1)x = (1-1)x = 0x = 0.$$

Then both parts of this problem become a simple matter of using this fact and and the distributive law.

However, Wolf does not require the existence of a multiplicative identity in his definition of a ring. We can still do the problem, mimicking the trick we just used with -1. To show that $(-x) + (-y)$ is the additive inverse of $x + y$, we need to check that

$$\begin{aligned}(x + y) + ((-x) + (-y)) &= x + y + (-x) + (-y) \\ &= (x + (-x)) + (y + (-y)) \\ &= 0 + 0 \\ &= 0.\end{aligned}$$

The proof for part (b) is similar.

Problem W 8.1.3

Here I give only the answers. Remember that for your problem sets, unless explicitly stated otherwise, you need to justify and explain your answers. Don't be dismayed if, as in this problem, this seems to mean that you have to repeatedly verify lots of annoying axioms. If you think for a bit there's usually a way to avoid such tedious mindless work by one or two incisive observations—that's what mathematics is all about. In this case, if you just "happened" to chose four problems where the example structure was *not* a ring then all you needed to do was to point out one ring axiom that failed. If it is obvious that a particular axiom is satisfied then its enough to say so (but do say it, don't just not say nothing), but make sure that you do address those axioms that are not obviously

Part (a) A ring without identity.

Part (b) Not a ring: distributivity fails.

Part (c) Not a ring: additive associativity fails.

Part (d) This is a field, which is "essentially the same as" (*i.e.* isomorphic to) the reals with the normal operations.

Part (e) This is a ring with additive identity $(0, 0)$ and multiplicative identity $(1, 1)$. Sometimes you will therefore see strange things like

$$\begin{aligned}0 &= (0, 0) \\ 1 &= (1, 1)\end{aligned}$$

which really means

$$\begin{aligned}0_{\mathbb{R}\times\mathbb{R}} &= (0_{\mathbb{R}}, 0_{\mathbb{R}}) \\ 1_{\mathbb{R}\times\mathbb{R}} &= (1_{\mathbb{R}}, 1_{\mathbb{R}})\end{aligned}$$

where 0_R just means the additive identity in the ring R and similarly, 1_R just means the multiplicative identity in the ring R. If the context is understood, we simply write 0 and 1.

However, this is not a field since non-zero elements such as $(0,1)$ and $(1,0)$ have no multiplicative inverses. In general, these very same two elements show that the cross product (*a.k.a.* direct product) of two fields is never a field, since every field contains 0 and 1, and these two elements are always non-zero and never have (multiplicative) inverses.

Part (f) This is a ring: all the axioms go work because they work *pointwise* on the functions. It's not a field since only functions that are never zero have multiplicative inverses.

Part (g) Not a ring: distributivity fails.

Part (h) This is a ring with identity, but not a field.

Problem W 8.1.5

Part (a) Again here, when the problem says determine, you have to justify what you say and explain. All you need to remark for most of the axioms (*i.e.* axioms $V3-V7$ and $V12-V17$) is that they hold because they do in the entirety of $\mathbb{R}$. Then it remains to show that axioms $V1, V2, V9$ (pretty clear) and $V11$ still hold, while axioms $V8$ and $V10$ fail.

Part (b) There are many right answers. The set of non-negative reals is one. $\mathbb{Q}$-$\{0\}$ is another.

Problem W 8.1.10 and 8.1.11

Recall that "the smallest subset of X satisfying the predicate P" is a subset $A \subseteq X$ satisfying P such that for every $B \subseteq X$ satisfying P, $A \subseteq B$. By this definition, as we saw in class, if such a smallest element exists then it must be the intersection of all $B \subseteq X$ satisfying P. But the most important part is that if the intersection of all B satisfying P does not also satisfy P there is no smallest element satisfying P (since if there were it would have to be that intersection).

In the first problem (8.1.10) our predicate is "is a closed interval containing 3,7 and 15." The intersection at hand is

$$A = \bigcap_{a \leq 3,\, 15 \leq b} [a, b].$$

Since $[3,15] \subseteq [a,b]$ in every case, $[3,15] \subseteq A$ also. And since $[3,15]$ is infact one of the intersected closed intervals, $A \subseteq [3,15]$ also, proving that $A = [3,15]$, which is in fact a closed interval containing $3, 7$ and 15, so it is our desired smallest such set.

In the second problem (8.1.11) our predicate is "is an open interval containing $3, 7$ and 15." The intersection at hand is

$$B = \bigcap_{a<3,\, 15<b} (a, b).$$

Again since $[3, 15] \subseteq [a, b]$ in every case, $[3, 15] \subseteq B$ also. This time $[3, 15]$ is not one of the intersected sets so we can't make the same argument as before. Instead we note that for $x \notin [3, 15]$ we know that for $0 < \epsilon < \min\{|3-x|, |15-x|\}$

$$x \notin (3 - \epsilon, 15 + \epsilon),$$

so $x \notin B$. Thus, $B = [3, 15]$. But this is not an open interval and thus does not satisfy our predicate. We conclude that there is no smallest open interval containing $3, 7$ and 15.