Math 101
Problem Set 1 Solutions

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Problem N 1.3.1

(a) $\mathcal{K}A = \{x \in \mathbb{Z} : \exists\ a \in A \text{ s.t. } a < x\}$. Not a closure operator:

$$\{0\} \not\subseteq \mathcal{K}\{0\} = \{1, 2, 3, \dots\}.$$

(b) $\mathcal{K}A = \{x \in \mathbb{Z} : \exists\ a \in A \text{ s.t. } a \leq x\}$. A closure operator:

C1. If A has no minimum then $\mathcal{K}A = \mathbb{Z}$ which contains A. If A has a minimum, a_0, then $\mathcal{K}A = \{x \in \mathbb{Z} : a_0 \leq x\}$ which also contains A.

C2. If either A or B has no minimum (or both) then

$$\mathcal{K}A \cup \mathcal{K}B = \mathcal{K}(A \cup B) = \mathbb{Z}$$

so we're done. If both A and B have minima, a_0 and b_0, respectively

$$\begin{aligned}\mathcal{K}(A \cup B) &= \{x \in \mathbb{Z} : \min\{a_0, b_0\} \leq x\} \\ &= \{x \in \mathbb{Z} : a_0 \leq x\} \cup \{x \in \mathbb{Z} : b_0 \leq x\} \\ &= \mathcal{K}A \cup \mathcal{K}B.\end{aligned}$$

C3. If A has no minimum then $\mathcal{K}\mathcal{K}A = \mathcal{K}A = \mathbb{Z}$. If it has a minimum, a_0 then $\mathcal{K}\mathcal{K}A$ and $\mathcal{K}A$ both equal

$$\{x \in \mathbb{Z} : a_0 \leq x\}.$$

C4. $\mathcal{K}\emptyset = \emptyset$ is clear.

(c) $\mathcal{K}A = \{x \in \mathbb{Z} : \exists\ a \in A, k \in \mathbb{Z} \text{ s.t. } x = ka\}$. A closure operator.
For $A, B \subseteq \mathbb{Z}$ denote $AB = \{x \in \mathbb{Z} : x = yz \text{ for some } y, z \in \mathbb{Z}\}$. Observe that

$$\mathbb{Z}\mathbb{Z} = \mathbb{Z} \tag{1}$$
$$\mathcal{K}A = \mathbb{Z}A \tag{2}$$
$$(AB)C = A(BC) \tag{3}$$

for $A, B, C \subseteq \mathbb{Z}$.

C1. Use observation (2): $\mathcal{K}A = \mathbb{Z}A \supseteq \{1\}A = A$.

C2. Calculate using observation (2) again

$$\begin{aligned}\mathcal{K}(A \cup B) &= \mathbb{Z}(A \cup B) \\ &= \{y \in \mathbb{Z} : y = nx \text{ for some } n \in \mathbb{Z}, x \in A \cup B\} \\ &= \{y \in \mathbb{Z} : y = nx \text{ for some } n \in \mathbb{Z}, x \in A\} \cup \\ &\quad \{y \in \mathbb{Z} : y = nx \text{ for some } n \in \mathbb{Z}, x \in B\} \\ &= \mathbb{Z}A \cup \mathbb{Z}B \\ &= \mathcal{K}A \cup \mathcal{K}B\end{aligned}$$

C3. Using observations (1) through (3) we get

$$\mathcal{K}\mathcal{K}A = \mathcal{K}(\mathbb{Z}A) = \mathbb{Z}(\mathbb{Z}A) = (\mathbb{Z}\mathbb{Z})A = \mathbb{Z}A = \mathcal{K}A.$$

C4. $\mathcal{K}\emptyset = \emptyset$ is clear.

(d) $\mathcal{K}A = \{x \in \mathbb{Z} : \exists\, a_1, a_2 \in A \text{ s.t. } a_1 \le x \le a_2\}$. Not a closure operator:

$$\begin{aligned}&\mathcal{K}\{1\} \cup \mathcal{K}\{3\} = \{1\} \cup \{3\}\,, \text{ but} \\ &\mathcal{K}(\{1\} \cup \{3\}) = \mathcal{K}\{1,3\} = \{1,2,3\}.\end{aligned}$$

Problem N 1.3.2

Check that $(L, \mathcal{K})$ satisfies the Kuratowski Axioms:

C1. Since both $A \subseteq A$ and $A \subseteq A \cup \{\infty\}$, $A \subseteq \mathcal{K}A$ for all A.

C2. If both A and B are finite then $A \cup B$ is finite too so

$$\mathcal{K}(A \cup B) = A \cup B = \mathcal{K}A \cup \mathcal{K}B.$$

Otherwise, say A is infinite, and thus so is $A \cup B$. Then

$$\mathcal{K}A \cup \mathcal{K}B = A \cup B \cup \{\infty\} = \mathcal{K}(A \cup B)$$

C3. If A is finite we have $\mathcal{K}A = A \implies \mathcal{K}\mathcal{K}A = \mathcal{K}A = A$. Otherwise we have

$$\mathcal{K}\mathcal{K}A = \mathcal{K}(A \cup \{\infty\}) = \mathcal{K}A \cup \{\infty\} = A \cup \{\infty\} = \mathcal{K}A.$$

C4. $\mathcal{K}\emptyset = \emptyset$ is clear.

Problem N 1.4.1

(a) Apply axiom **C2.** twice:

$$\mathcal{K}(A_1 \cup A_2 \cup A_3) = \mathcal{K}(A_1 \cup A_2) \cup \mathcal{K}A_3 = \mathcal{K}A_1 \cup \mathcal{K}A_2 \cup \mathcal{K}A_3$$

(b) Apply axiom **C2.** three times. (It's the same.)

(c) The general conjecture is that for $n \in \mathbb{Z}$

$$\mathcal{K}\bigcup_{k=1}^{n} A_k = \bigcup_{k=1}^{n} \mathcal{K}A_k.$$

We could simply say here that we apply axiom **C2.** $n - \mathbf{1}$ times but that would be glossing over something very important which is happening here—induction. We've already done the base case and now we simply observe that

$$\mathcal{K}\bigcup_{k=1}^{n} A_k = \mathcal{K}\bigcup_{k=1}^{n-1} A_k \cup A_n$$

and by the *induction hypothesis* (= the assumption that what you'd like to prove is true for smaller n) we already know that

$$\mathcal{K}\bigcup_{k=1}^{n-1} A_k = \bigcup_{k=1}^{n-1} \mathcal{K}A_k,$$

so we're done.

If we replace unions with intersections then the resulting conjecture is blatantly false already in the case $n = 2$. (Consider *e.g.* on the real line $\mathbb{R}$ with the standard Euclidean topology: $\mathcal{K}([0,1) \cap (1,2]) = \mathcal{K}(\emptyset) = \emptyset$ but $\mathcal{K}[0,1) \cap \mathcal{K}(1,2] = [0,1] \cap [1,2] = \{1\}$.) However, Theorem N 1.10 says that we do have $\mathcal{K}(A \cap B) \subseteq \mathcal{K}A \cap \mathcal{K}B$ and using induction as before we get the more general result

$$\mathcal{K}\bigcap_{k=1}^{n} A_k \subseteq \bigcap_{k=1}^{n} \mathcal{K}A_k.$$

Problem N 1.4.2

(a) Let $X = \mathbb{Z}$, let $\mathcal{K}$ be the closure operator defined in Problem N 1.3.1(b) (which we showed was in fact a closure operator) and let $A = \{1\}$ and $B = \{2\}$. Then $\mathcal{K}A = \{1, 2, 3, \dots\}$ and $\mathcal{K}B = \{2, 3, 4, \dots\}$ so

$$\mathcal{K}(A - B) = \mathcal{K}A = \{1, 2, 3, \dots\} \not\subseteq \{1\} = \mathcal{K}A - \mathcal{K}B.$$

(b) Begin with the following lemma:

Lemma. $C - D \subseteq E \iff C \subseteq D \cup E$.

Proof. Suppose $C - D \subseteq E$. If $x \in C$ then $x \in E$ or $x \in D$ (and $x \notin C - D$). This in turn implies $x \in D \cup E$. So, $C \subseteq D \cup E$.

Now, if $C - D \not\subseteq E$, then there exists $x \in C - D$ such that $x \notin E$. Furthermore, $x \in C$ but $x \notin D$ implies $x \notin D \cup E$ and finally $C \not\subseteq D \cup E$. □

Now clearly $A \subseteq A \cup B = B \cup (A - B)$. By Theorem N 1.9 (that $C \subseteq D$ implies $\mathcal{K}C \subseteq \mathcal{K}D$), we therefore have that

$$\mathcal{K}A \subseteq \mathcal{K}(B \cup (A - B)) = \mathcal{K}B \cup \mathcal{K}(A - B),$$

which—by the Lemma—happens if and only if

$$\mathcal{K}A - \mathcal{K}B \subseteq \mathcal{K}(A - B).$$

Problem N 1.4.3

Recall the following definitions.

$$\begin{aligned} \operatorname{Ext} A &\stackrel{\text{def}}{=} (\mathcal{K}A)^c = X - \mathcal{K}A \\ \operatorname{Int} A &\stackrel{\text{def}}{=} (\mathcal{K}A^c)^c = X - \mathcal{K}(X - A) \\ \partial A &\stackrel{\text{def}}{=} \mathcal{K}A \cap \mathcal{K}A^c = \mathcal{K}A \cap \mathcal{K}(X - A) \end{aligned}$$

For brevity we denote the complement of a set S by $S^c = X - S$.

(a) There are four properties to verify:

$$X = \operatorname{Ext} A \cup \operatorname{Int} A \cup \partial A \tag{4}$$
$$\emptyset = \operatorname{Ext} A \cap \operatorname{Int} A \tag{5}$$
$$\emptyset = \operatorname{Ext} A \cap \partial A \tag{6}$$
$$\emptyset = \operatorname{Int} A \cap \partial A. \tag{7}$$

First we verify (4) using De Morgan's Law repeatedly and cancelling as we go:

$$\begin{aligned} \operatorname{Ext} A \cup \operatorname{Int} A \cup \partial A &= (\mathcal{K}A)^c \cup (\mathcal{K}A^c)^c \cup (\mathcal{K}A \cap \mathcal{K}A^c) \\ &= (\mathcal{K}A)^c \cup \Big[\big[(\mathcal{K}A^c)^c \cup \mathcal{K}A \big] \cap \big[(\mathcal{K}A^c)^c \cup \mathcal{K}A^c \big] \Big] \\ &= (\mathcal{K}A)^c \cup \Big[\big[(\mathcal{K}A^c)^c \cup \mathcal{K}A \big] \cap X \Big] \\ &= (\mathcal{K}A)^c \cup (\mathcal{K}A^c)^c \cup \mathcal{K}A \\ &= X \cup (\mathcal{K}A^c)^c \\ &= X. \end{aligned}$$

Next we verify (5) by noting that for any set $B \subseteq X$ we have

$$(\mathcal{K}B^c)^c \subseteq B.$$

Using this twice, first with $B = A$ and second with $B = A^c$, we get

$$\begin{aligned} \operatorname{Ext} A \cap \operatorname{Int} A &= (\mathcal{K}A)^c \cap (\mathcal{K}A^c)^c \\ &\subseteq (\mathcal{K}A)^c \cap A \\ &\subseteq A^c \cap A \\ &= \emptyset. \end{aligned}$$

Finally we verify (6) and (7) by simply cancelling in the obvious fashion:

$$\begin{aligned} \operatorname{Ext} A \cap \partial A &= (\mathcal{K}A)^c \cap \mathcal{K}A \cap \mathcal{K}A^c = \emptyset, \\ \operatorname{Int} A \cap \partial A &= (\mathcal{K}A^c)^c \cap \mathcal{K}A \cap \mathcal{K}A^c = \emptyset. \end{aligned}$$