2

a. This is not always true. If $A = (0,2)$ and $B = (0,1)$, then $\mathbf{K}(A - B) = \mathbf{K}([1,2)) = [1,2]$, but $\mathbf{K}A - \mathbf{K}B = [0,2] - [0,1] = (1,2]$. Of course $[1,2]$ is not contained in $(1,2]$, so this is a counterexample.

b. We first prove the following lemma.

Lemma 1. *$C - D \subset E$ if and only if $C \subset D \cup E$*

Proof. ($\Rightarrow$) Assume that $C - D \subset E$. Let $x \in C$. If $x \in D$, then $x \in D \cup E$. If $x \notin D$, then $x \in C - D \subset E$ so $x \in D \cup E$. Thus $C \subset D \cup E$ as desired.

($\Leftarrow$) Assume $C \subset D \cup E$. Let $x \in C - D$. We know $x \in C \subset D \cup E$ so x is in D or E. By hypothesis, $x \notin D$, so $x \in E$. Thus $C - D \subset E$ as desired. □

By the Lemma, it suffices to show that $\mathbf{K}A \subset \mathbf{K}(A - B) \cup \mathbf{K}B$. We know $\mathbf{K}(A - B) \cup \mathbf{K}B = \mathbf{K}((A - B) \cup B) = \mathbf{K}(A \cup B)$. Clearly $A \subset A \cup B$, so by the Lemma from class, $\mathbf{K}A \subset \mathbf{K}(A \cup B) = \mathbf{K}(A - B) \cup \mathbf{K}B$, and our proof is complete.

3

a. We must first show that the union of these three sets is all of X. By definition $ExtA \cup IntA \cup \delta A = (X - \mathbf{K}A) \cup (X - \mathbf{K}(X - A)) \cup (\mathbf{K}A \cap \mathbf{K}(X - A)) = (X - (\mathbf{K}A \cap \mathbf{K}(X - A)) \cup (\mathbf{K}A \cap \mathbf{K}(X - A)) = X$. Thus, every point in X is in at least one of $ExtA, IntA$, and δA. It remains to show that these sets are disjoint. For this we note that $ExtA \cap \delta A = (X - \mathbf{K}A) \cap (\mathbf{K}A \cap \mathbf{K}(X - A)) = X \cap (\mathbf{K}A)^c \cap \mathbf{K}A \cap \mathbf{K}(X - A) = ((\mathbf{K}A)^c \cap \mathbf{K}A) \cap X \cap \mathbf{K}(X - A) = \emptyset \cap X \cap \mathbf{K}(X - A) = \emptyset$. Similarly $IntA \cap \delta A = (X - \mathbf{K}(X - A)) \cap (\mathbf{K}A \cap \mathbf{K}(X - A)) = X \cap (\mathbf{K}(X - A))^c \cap \mathbf{K}A \cap \mathbf{K}(X - A) = ((\mathbf{K}(X - A))^c \cap \mathbf{K}(X - A)) \cap X \cap \mathbf{K}A = \emptyset \cap X \cap \mathbf{K}A = \emptyset$. Finally, $ExtA \cap IntA = (X - \mathbf{K}A) \cap (X - \mathbf{K}(X - A)) = X - (\mathbf{K}A \cup \mathbf{K}(X - A))$. As $A \subset \mathbf{K}A$ and $X - A \subset \mathbf{K}(X - A)$, $\mathbf{K}A \cup \mathbf{K}(X - A) \supset A \cup (X - A) = X$. So $ExtA \cap IntA = X - X = \emptyset$. Thus the union is disjoint as required.

b. By definition, $A \cup \delta A = A \cup (\mathbf{K}A \cap \mathbf{K}(X - A)) = (A \cup \mathbf{K}A) \cap (A \cup \mathbf{K}(X - A))$ by De Morgan's laws. Clearly $A \cup \mathbf{K}A = \mathbf{K}A$ as $\mathbf{K}A \supset A$. Also $X - A \subset \mathbf{K}(X - A)$ so $A \cup \mathbf{K}(X - A) = X$. Hence, $A \cup \delta A = \mathbf{K}A \cap X = \mathbf{K}A$ as desired.

c. By definition $A - \delta A = A - (\mathbf{K}A \cap \mathbf{K}(X - A)) = (A - \mathbf{K}A) \cup (A - \mathbf{K}(X - A))$. As $A \subset \mathbf{K}A$, $A - \mathbf{K}A = \emptyset$. So $A - \delta A = \emptyset \cup (A - \mathbf{K}(X - A)) = A - \mathbf{K}(X - A)$. Clearly this is a subset of $X - \mathbf{K}(X - A) = IntA$. For the converse, we note that $IntA = X - \mathbf{K}(X - A) \subset X - (X - A) = A$ as $(X - A) \subset \mathbf{K}(X - A)$. So $IntA \subset A$. But by part a, $IntA \cap \delta A = \emptyset$, so $IntA \subset A - \delta A$. This shows that these sets are equal, as desired.

4

a. We note that $IntA = X - \mathbf{K}(X - A) \subset X - (X - A) = A$ as $(X - A) \subset \mathbf{K}(X - A)$. So $IntA \subset A$.

b. By definition $IntA \cap IntB = (X - \mathbf{K}(X - A)) \cap (X - \mathbf{K}(X - B)) = X - (\mathbf{K}(X - A) \cup \mathbf{K}(X - B))$ by De Morgan's laws. By Axiom 2, $\mathbf{K}(X - A) \cup \mathbf{K}(X - B) = \mathbf{K}((X - A) \cup (X - B)) = \mathbf{K}(X - (A \cap B))$, by De Morgan's laws again. So $IntA \cap IntB = X - \mathbf{K}(X - (A \cap B)) = Int(A \cap B)$ as desired.

c. By definition $IntIntA = X - \mathbf{K}(X - (X - \mathbf{K}(X - A))) = X - \mathbf{K}(\mathbf{K}(X - A))$. By Axiom 3, $\mathbf{K}\mathbf{K}(X - A) = \mathbf{K}(X - A)$, so $IntIntA = X - \mathbf{K}(X - A) = IntA$ as desired.

d. Finally, $IntX = X - \mathbf{K}(X - X) = X - \mathbf{K}(\emptyset) = X - \emptyset = X$, using Axiom 4.