2

Clearly for each $i \in I$, $A_i \subset \cup i \in IA_i$. So by the lemma in Goroff, $\mathbf{K}A_i \subset \mathbf{K} \cup_{i\in I} A_i$. As this is true for each $i \in I$, $\mathbf{K} \cup_{i\in I} A_i$ contains the union of the $\mathbf{K}A_i$ and $\cup_{i\in I}\mathbf{K}A_i \subset \mathbf{K} \cup_{i\in I} A_i$.

Clearly for each $i \in I$, $\cap_{i\in I}A_i \subset A_i$. So by the lemma in Goroff, $\mathbf{K} \cap_{i\in I} \mathbf{K}A_i \subset \mathbf{K}A_i$. As this is true for each $i \in I$,$\mathbf{K} \cap_{i\in I} \mathbf{K}A_i$ is contained in the intersection of the $\mathbf{K}A_i$ and $\mathbf{K} \cap_{i\in I} \mathbf{K}A_i \subset \cap_{i\in I}\mathbf{K}A_i$.

3

I was a bit flexible here because it was not clear what topology was intended for $\mathbb{Z}^+$. Under the Euclidean topology (inherited as a subset of $\mathbb{R}$), each point in $\mathbb{Z}^+$ is closed, because there is nothing "close" to it (in the Euclidean) sense. In fact, every subset of $\mathbb{Z}^+$ is closed because there are no integers close to a set of integers that do not belong to that set. But this means that every set is open because it's complement is closed. So every subset of $\mathbb{Z}^+$ is clopen. For $\mathbf{L}$, every finite set is closed (because it is its own closure), as is every infinite set that includes the element ∞. Thus, finite sets, which do not include infinite are open (for the complement will be an infinite set including infinity). Thus finite sets which do not include the element ∞ are clopen. It is clear that the complements of finite sets that do not include the element ∞ must be clopen as well. Naturally $\emptyset$ and $\mathbf{L}$ are clopen as well. I claim that this is it. Given a set A not of this form, either A is finite but does include ∞, in which case its complement is not closed, so this set is not open. Otherwise the set A is infinite. In this case, either A does not include ∞, in which case it is not closed, or it does, in which case, it's complement is infinite and does not include ∞, so is not closed. So the only clopen sets of $\mathbf{L}$ are $\mathbf{L}$, $\emptyset$, finite sets including ∞, and infinite sets with finite complements that do include ∞.

In general, a space in which every subset is both open and closed has the discrete topology ($\mathbf{K}A = A$ for all $A \subset X$). To see this, note that if every subset is clopen then every subset must be closed, so $\mathbf{K}A = A$.

4

Clearly the decimal $0.a_1a_2a_3\dots$ is less than or equal to the decimal that replaces each digit by a 9 (to see this, replace the digits one at a time). So we must show that $0.999\dots = 9/10 + 9/100 + 9/1000 + \cdots = \sum_{i=1}^{\infty} \frac{9}{10^i}$ is finite and never exceeds 1. By the geometric series formula (proven previously), this sum equals

$$\frac{\frac{9}{10}}{1 - \frac{1}{10}} = \frac{9}{10 - 1} = 1.$$

As $0.a_1a_2\dots$ is less than or equal to this, we are done.

5

This was done in class. Elements of the Cantor set are precisely those real numbers expressible as an infinte decimal with all digits 0 or 2 in base 3. To see this, note that the n-th digit of an element of the Cantor set is determined by which interval that element is in in A_n.

6

I will give two proofs of this fact. First, using problem 5, the elements of the Cantor set are precisely those real numbers that can be expressed in the form $0.a_1a_2a_3\cdots$ where each $a_i \in \{0, 2\}$. If the Cantor set is countable, we can label these elements $x_1, x_2, \dots$ etc. We then define z to be the decimal $0.b_1b_2\dots$ where $b_i = 0$ if the i-th digit of x_i is 2 and vice versa. If the Cantor set is countable, then z must equal some x_i, but by construction the i-th digits differ so these numbers are distint. Thus, the Cantor set is uncountable.

Alternatively, we establish a bijection between the Cantor set and the power set of $\mathbb{N}$, which we know to be uncountable. By definition two sets have the same cardinality if there is a bijection between them so this would show that the Cantor set is uncountable as well. We construct our bijection as follows. Given x in the Cantor set, we must have $x \in \cap_{i\in\mathbb{N}} A_i$. We examine what this means. Given $x \in A_n$, x is in some closed interval of length $\frac{1}{3^n}$. If $x \in A_{n+1}$, then x is either in the first third of this interval, or in the last third. We now associate with x in the Cantor set an infinite string of 0s and 1s, which we know is in bijection with the power set of $\mathbb{N}$ (recall a 0 in the n-th spot indicates that n is not in a particular subset and a 1 means that it is). Because x is in the Cantor set, $x \in A_1$ so x is in either the left or right interval. If x is in the left, put a 0 in the first place; if it is the right put a 1. Now x must also be in A_2 by the argument above, so x is either in the left interval that results from removing the middle third from the interval containing x in A_1 or it is in the right. If x is in the left, put a 0 in the 2nd place. Otherwise put a 1. Proceed like this.

We must now show that this map is a bijection. If $x \neq y$ are both in the Cantor set, then there must be some digit in which they differ, i.e., some A_n such that x is in the left interval and y is in the right. So the subsets corresponding to x and y are different and this map is one-to-one. To show that it is onto, we claim that any infinite string of 0s and 1s gives an element in the Cantor set. The best way to see this is to note that by changing the 1s to 2s and the infinite string to an infinite decimal, we get an element of the Cantor set (by problem 5) that is precisely the thing that maps to this subset of $\mathbb{N}$. So our map is onto, and we do in fact have a bijection.