Math S–101. Assignment 1.
Due Wednesday, July 5, 2006

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1. Prove that
$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$
for $n \in \mathbb{N}$.

2. Prove that $4 \cdot 10^{2n} + 9 \cdot 10^{2n-1} + 5$ is divisible by 99 for $n \in \mathbb{N}$.

3. **Fibonacci Numbers.** The Fibonacci numbers are
$$1, 1, 2, 3, 5, 8, 13, 21, \ldots.$$
We can define them inductively by $f_1 = 1$, $f_2 = 1$, and $f_{n+2} = f_{n+1} + f_n$ for $n \in \mathbb{N}$.

   (a) Prove that $f_n < 2^n$.

   (b) Prove that $f_{n+1}f_{n-1} = f_n^2 + (-1)^n$, $n \geq 2$.

   (c) Prove that $f_n = [(1+\sqrt{5}\,)^n - (1-\sqrt{5}\,)^n]/2^n\sqrt{5}$.

   (d) Show that $\lim_{n\to\infty} f_n/f_{n+1} = (\sqrt{5}-1)/2$.
   If you are unfamiliar with the concept of a limit, please ask.

   (e) Prove that f_n and f_{n+1} are relatively prime.

4. Suppose that a, b, r, s are coprime and that
$$\begin{aligned} a^2 + b^2 &= r^2 \\ a^2 - b^2 &= s^2. \end{aligned}$$
Prove that a, r, and s are odd and b is even.

5. Define the *least common multiple* of two nonzero integers a and b, denoted by $\operatorname{lcm}(a, b)$, to be the nonnegative integer m such that both a and b divide m, and if a and b divide any other integer n, then m also divides n. Prove that any two integers a and b have a unique least common multiple.

6. If $d = \gcd(a, b)$ and $m = \operatorname{lcm}(a, b)$, prove that $dm = |ab|$.

7. Show that $\operatorname{lcm}(a, b) = ab$ if and only if $\gcd(a, b) = 1$.

8. Let $p \geq 2$. Prove that if $2^p - 1$ is prime, then p must also be prime.

9. Prove that there are an infinite number of primes of the form $4n - 1$.

10. Using the fact that 2 is prime, show that there do not exist integers p and q such that $p^2 = 2q^2$. Demonstrate that therefore $\sqrt{2}$ cannot be a rational number.

11. **Challenge Problem**.[1] Show that

$$\sqrt[n]{a_1 a_2 \cdots a_n} \leq \frac{1}{n} \sum_{k=1}^{n} a_k.$$

[1]Let $A_n = (a_1 + \cdots + a_n)/n$ and $G_n = \sqrt[n]{a_1 \cdots a_n}$. If $n = 2$, then $a_1 - 2\sqrt{a_1 a_2} + a_2 = (\sqrt{a_1} - \sqrt{a_2})^2 \geq 0$. Therefore, $(a_1 + a_2)/2 \geq \sqrt{a_1 a_2}$. The next step is to prove the inequality for $n = 2^k$ by induction on k. Now let $2^m \geq n$. Now apply the last step to the 2^m numbers

$$a_1, \cdots, a_n, \underbrace{A_n, \cdots, A_n}_{2^m - n \text{ times}}$$

to show that $G_n \leq A_n$ for n in general