Math S–101. Final Exam. Solutions.

T. Judson

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1. Let $f : X \to Y$ be a well-defined map. Let A and B be subsets of Y. Prove that $f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B)$.

   Solution.

$$
\begin{aligned}
x \in f^{-1}(A - B) &\iff f(x) \in A - B \\
&\iff f(x) \in A \text{ and } f(x) \notin B \\
&\iff x \in f^{-1}(A) \text{ and } x \notin f^{-1}(B) \\
&\iff x \in f^{-1}(A) - f^{-1}(B).
\end{aligned}
$$

2. Let X and Y be topological spaces and $f : X \to Y$ be a well-defined map. Prove that the following two statements are equivalent.

   (a) For each open set $U \subset Y$, the set $f^{-1}(U)$ is open in X.

   (b) For each closed set $E \subset Y$, the set $f^{-1}(E)$ is closed in X.

   Hint: You may assume Problem 1 is true in your proof.

   Solution. $(a) \implies (b)$ Suppose that E is a closed subset of Y. Then $U = Y - E$ is an open subset of Y. Thus, $f^{-1}(U)$ is open in X and

$$f^{-1}(E) = f^{-1}(Y - U) = f^{-1}(Y) - f^{-1}(U) = X - f^{-1}(U).$$

   is closed.

   $(b) \implies (a)$ Conversely, assume that set $f^{-1}(E)$ is closed in X for each closed set E contained in Y. If U is any open set contained in Y, then $E = Y - U$ is closed. Therefore,

$$f^{-1}(U) = f^{-1}(Y - E) = f^{-1}(Y) - f^{-1}(E) = X - f^{-1}(E).$$

   must be open.

3. Let $(X, \mathbf{K}_X)$ and $(Y, \mathbf{K}_Y)$ be topological spaces. Suppose that X is connected and $f : X \to Y$ is continuous with $f(X) = Y$. Prove that Y is connected. *Hint*: Assume that Y is not connected.

Solution. Assume that Y is not connected and A and B separate Y. Then A and B are nonempty, closed subsets of Y such that $Y = A \cup B$ and $A \cap B = \emptyset$. We also know that

$$f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = f^{-1}(Y) = X$$

and

$$f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\emptyset) = \emptyset,$$

Since $f(X) = Y$ and A and B are nonempty, $f^{-1}(A)$ and $f^{-1}(B)$ are nonempty. These two sets are also closed since f is continuous. Therefore, $f^{-1}(A)$ and $f^{-1}(B)$ separate X, which is a contradiction.

4. If $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and $c \in \mathbb{R}$, prove that the function $c \cdot f$ defines a continuous function from $\mathbb{R}^n$ to $\mathbb{R}$. Be sure to consider the case $c = 0$.

   Solution. Let $x \in \mathbb{R}$ and $\epsilon > 0$. We must find a $\delta > 0$ such that for all $y \in \mathbb{R}^n$,

$$d(x, y) < \delta \Longrightarrow |cf(x) - cf(y)| < \epsilon.$$

   Since f is continuous, we can choose $\delta > 0$ such that

$$d(x, y) < \delta \Longrightarrow |f(x) - f(y)| < \frac{\epsilon}{1 + |c|}.$$

   Then

$$d(x, y) < \delta \Longrightarrow |cf(x) - cf(y)| = |c| \cdot |f(x) - f(y)| < \frac{|c|\epsilon}{1 + |c|} < \epsilon.$$

   We use $\epsilon/(1 + |c|)$ instead of $\epsilon/|c|$ to avoid dividing by zero in the case that $c = 0$.

5. Let $X = \mathbb{R}$ with the Euclidean closure operator $\mathbf{K}$. Prove that the closure of the interval (a, ∞) is $[a, \infty)$.

   Solution. We know that $\mathbf{K}(a, \infty) \subset \mathbb{R}$. We must show that $x \in \mathbf{K}(a, \infty)$ if and only if $x \in [a, \infty)$. We will consider three cases.

   - Let $x < a$ and $\epsilon = a - x$. Then $B(x; \epsilon) \cap (a, \infty) = \emptyset$ Thus, $x \notin \mathbf{K}(a, \infty)$.
   - Let $x = a$ and $\epsilon > 0$. Then $B(x; \epsilon) \cap (a, \infty) \neq \emptyset$. Since this is true for all $\epsilon > 0$, $x \in \mathbf{K}(a, \infty)$.
   - Let $x > a$. Since $\mathbf{K}(a, \infty) \supset (a, \infty)$, we know that $x \in \mathbf{K}(a, \infty)$

   Therefore, the closure of the interval (a, ∞) is $[a, \infty)$.

6. Prove that $\sqrt{6}$ is irrational.

   Solution. Assume that $\sqrt{6}$ is irrational and that $\sqrt{6} = p/q$, where p and q are relatively prime. Then $6q^2 = p^2$ and p must be even. Therefore, we can write $p = 2r$ for some number r and

$$6q^2 = p^2 = 4r^2 \qquad \text{or} \qquad 3q^2 = 2r^2.$$

   Thus, q must also be even, which contradicts the fact that p and q are relatively prime.

7. Let $(X, \mathbf{K}_X)$, $(Y, \mathbf{K}_Y)$, and $(Z, \mathbf{K}_Z)$ be topological spaces. Suppose that $f : X \to Y$ and $g : Y \to Z$ are continuous. Prove that the composition of f and g,

$$g \circ f : X \to Z,$$

must also be continuous.

Solution. Let U be an open subset of Z. Then $g^{-1}(U)$ is open in Y and

$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$$

is open in X. Therefore, $g \circ f : X \to Z$ is continuous.

Alternate Solution. Suppose that $A \subset X$. We must show that $gf\,\mathbf{K}_X\,A \subset \mathbf{K}_Z\,gfA$. Since f is continuous, $f\,\mathbf{K}_X\,A \subset \mathbf{K}_Y\,f(A)$. Using the facts that g is continuous and $A \subset B \implies f(A) \subset f(B)$, we have

$$gf\,\mathbf{K}_X\,A \subset g\,\mathbf{K}_Y\,f(A) \subset \mathbf{K}_Z\,gf(A).$$

8. If two topological spaces $(X, \mathbf{K}_X)$ and $(Y, \mathbf{K}_Y)$ are homeomorphic, we write

$$(X, \mathbf{K}_X) \approx (Y, \mathbf{K}_Y).$$

Prove that $\approx$ is an equivalence relation. *Hint*: You may assume Problem 7 is true in your proof.

Solution. We will use the fact that $(X, \mathbf{K}_X)$ and $(Y, \mathbf{K}_Y)$ are homeomorphic if there exist continuous functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is the identity on X and $f \circ g$ is the identity on Y.

- If we define f (and g) to be the identity function that maps X to itself, then $(X, \mathbf{K}_X) \approx (X, \mathbf{K}_X)$ since the identity function is continuous and $g \circ f$ is the identity on X and $f \circ g$ is the identity on Y.
- Suppose that $(X, \mathbf{K}_X) \approx (Y, \mathbf{K}_Y)$. Then there exist continuous functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is the identity on X and $f \circ g$ is the identity on Y. Interchanging the roles of f and g, we have $(Y, \mathbf{K}_Y) \approx (X, \mathbf{K}_X)$.
- Suppose that $(X, \mathbf{K}_X) \approx (Y, \mathbf{K}_Y)$ and $(Y, \mathbf{K}_Y) \approx (Z, \mathbf{K}_Z)$. Then there exist continuous functions $f_1 : X \to Y$ and $g_1 : Y \to X$ such that $g_1 \circ f_1$ is the identity on X and $f_1 \circ g_1$ is the identity on Y, and continuous functions $f_2 : Y \to Z$ and $g_2 : Z \to Y$ such that $g_2 \circ f_2$ is the identity on Y and $f_2 \circ g_2$ is the identity on Z. Define $f = f_2 \circ f_1 : X \to Z$ and $g = g_1 \circ g_2 : Z \to X$. Both f and g are continuous by Problem 7. The function $g \circ f$ is the identity on X, since

$$(g \circ f)(x) = g(f(x)) = g_1(g_2(f_2(f_1(x))) = g_1(f_1(x)) = x.$$

  Similarly, $f \circ g$ is the identity on Y. Thus, $(X, \mathbf{K}_X) \approx (Z, \mathbf{K}_Z)$.

Alternate Solution. Two topological spaces $(X, \mathbf{K}_X)$ and $(Y, \mathbf{K}_Y)$ are homeomorphic if there exists a bijection $h : X \to Y$ such that for any $A \subset Y$,

$$\mathbf{K}_Y\,A = h\,\mathbf{K}_X\,h^{-1}A.$$

To show that $\approx$ is an equivalence relation, we will need to show that the reflexive, symmetric, and transitive properties hold.

- If $h : X \to X$ is the identity function, then h is a continuous bijection and

$$\mathbf{K}_X\, A = h\, \mathbf{K}_X\, h^{-1} A$$

for $A \subset X$. Thus, $(X, \mathbf{K}_X) \approx (X, \mathbf{K}_X)$.

- Suppose that $(X, \mathbf{K}_X) \approx (Y, \mathbf{K}_Y)$. Then there exists a continuous bijection $h : X \to Y$ such that for any $A \subset Y$,

$$\mathbf{K}_Y\, A = h\, \mathbf{K}_X\, h^{-1} A.$$

Define $g = h^{-1}$. Then for any $B \subset X$,

$$\mathbf{K}_X\, B = g\, \mathbf{K}_X\, g^{-1} B.$$

Thus, $(Y, \mathbf{K}_Y) \approx (X, \mathbf{K}_X)$.

- Suppose that $(X, \mathbf{K}_X) \approx (Y, \mathbf{K}_Y)$ and $(Y, \mathbf{K}_Y) \approx (Z, \mathbf{K}_Z)$. Then there exist continuous bijections $f : X \to Y$ and $g : Y \to X$ such for $A \subset Y$ and $B \subset Z$,

$$\begin{aligned} \mathbf{K}_Y\, A &= f\, \mathbf{K}_X\, f^{-1} A, \\ \mathbf{K}_Z\, B &= g\, \mathbf{K}_Y\, g^{-1} B. \end{aligned}$$

If we define $h = gf$ and use the fact that $h^{-1} = f^{-1} g^{-1}$, then for $B \subset Z$,

$$\mathbf{K}_Z\, B = g\, \mathbf{K}_Y\, g^{-1} B = gf\, \mathbf{K}_X\, f^{-1} g^{-1} B = h\, \mathbf{K}_X\, h^{-1} B.$$

Consequently, $(X, \mathbf{K}_X) \approx (Z, \mathbf{K}_Z)$.