Name:______________________________

Math S–101. Midterm 2—Tuesday, August 1, 2006

Problem Number	Possible Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total	50	

Please Read. This is a take home examination. Do any five of the following problems. If you do more than five problems, please indicate which problems that you would like us to read by crossing out the remaining solutions. If you fail to do so, only the first five problems in your exam booklet will be read. Any additional problems will be ignored even if they are correct.

Please be sure to write neatly in the exam booklet—illegible answers will receive little or no credit. Be sure to use correct mathematical notation. To receive full credit on a problem, you will need to justify your answers carefully—unsubstantiated answers will receive little or no credit.

Calculators are permitted, and you may use the course textbook, Prof. Goroff's notes, or your own notes as references; however, you are on your honor not to work with other students or use any other sources such as other textbooks or the Internet. Violation of these guidelines will be considered cheating. Please refer to pages 10–11 in the Summer School Student Handbook for the University's policy on Academic Honesty.

Good Luck!!!

1. Given a nonempty set X, show that

$$d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

defines a metric on X.

2. Define the Euclidean distance between two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$ to be
$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$
Show that d defines a metric on $\mathbb{R}^2$.

3. Consider the set $\mathbb{L} = \mathbb{Z}^+ \cup \{\infty\}$. Show that

$$d(n,m) = \begin{cases} |1/n - 1/m|, & \text{if } m, n \in \mathbb{Z}^+ \\ 1/n, & \text{if } m = \infty \end{cases}$$

defines a metric on $\mathbb{L}$.

4. Suppose that X and Y are disjoint sets. That is, $X \cap Y = \emptyset$. Suppose that $\mathbf{K}_X$ and $\mathbf{K}_Y$ are closure operators on X and Y, respectively. Define an operator $\mathbf{K}$ on $X \cup Y$ by

$$\mathbf{K}(A) = \mathbf{K}_X(A \cap X) \cup \mathbf{K}_Y(A \cap Y)$$

for each $A \subset X \cup Y$. Show that $\mathbf{K}$ is a closure operator.

5. Let X be a topological space with closure operator $\mathbf{K}$. If A is a connected subset of X, show that $\mathbf{K}(A)$ must be connected.

6. Let X be a topological space with closure operator $\mathbf{K}$. If A is connected and $A \subset B \subset \mathbf{K}(A)$, prove that B must also be connected.

7. Let $f : X \to Y$ be a well-defined map between two sets X and Y. Define the *inverse image* of a subset $A \subset Y$ under f to be the set

$$f^{-1}(A) = \{x \in X | f(x) \in A\}.$$

Notice that f^{-1} does not necessarily define a map from Y to X.

Now let X and Y be topological spaces and $f : X \to Y$ be a well-defined map. We say that f is *continuous* if for each open set $U \subset Y$, the set $f^{-1}(U)$ is open in X.

Show that f is a continuous function if and only if the set $f^{-1}(E)$ is closed in X for each closed set E contained in Y.

8. Suppose that A and B separate X and $Y \subset X$. If $A \cap Y \neq \emptyset$ and $B \cap Y \neq \emptyset$, show that $A \cap Y$ and $B \cap Y$ separate Y.