Assignment 1 Solution

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Problem 1

There are a few ways of doing this problem; here's one:
'$\Rightarrow$' If $gcd(a, c) = gcd(b, c) = 1$, then a and c have no common prime factors, and neither do b and c. If c has no prime factors in common with a or b, then it certainly cannot have any prime factors in common with ab. So $gcd(ab, c) = 1$
'$\Leftarrow$' If $gcd(ab, c) = 1$, then that means that c and ab have no prime factors in common. Hence c cannot have any prime factors in common with a or b. So $gcd(a, c) = gcd(b, c) = 1$.

Problem 2

We use the contrapositive: instead of proving "$2^p - 1$ prime $\Rightarrow$ p prime," we prove the logically equivalent statement "p not prime $\Rightarrow$ $2^p - 1$ not prime". Suppose p is not prime; then we can write $p = ab$ for some integers a and b greater than 1. Then we can write the following:

$$2^p - 1 = 2^{ab} - 1 = (2^a)^b - 1 = (2^a - 1)(1 + 2^a + 2^{2a} + ... + 2^{(b-1)a})$$

So now $2^p - 1$ is divisible by $2^a - 1$. Since $a > 1$, $2^a - 1$ is an integer greater than 1, so p is not prime.

Problem 3

The correct matches are:

- $P \to \sim Q$ is equivalent to $\sim (P \bigwedge Q)$
  As an example, the truth table for this one would look like this:

P	Q	$\sim Q$	$P \to \sim Q$	$P \bigwedge Q$	$\sim (P \bigwedge Q)$
T	T	F	F	T	F
T	F	T	T	F	T
F	T	F	T	F	T
F	T	F	T	F	T

- $P \leftrightarrow (P \bigwedge Q)$ is equivalent to $P \to Q$
- $(P \bigvee Q) \bigwedge \sim (P \bigwedge Q)$ is equivalent to $P \leftrightarrow \sim Q$
- $P \to \sim P$ is equivalent to $\sim P$
- $(P \bigvee Q) \leftrightarrow (P \bigwedge Q)$ is equivalent to $P \leftrightarrow Q$

Problem 4

a) "Pigs are fish or $2 + 2 \neq 4$." This is false, because both sides of the "or" are false.
b) "$2 + 2 = 4$ if and only if Canada is not in Asia." This is true, because both sides of the "if and only if" are true.
c) "If $2 + 2 \neq 4$, then Canada is in Asia and pigs are not fish." This is true, because "$2 + 2 \neq 4$" is false.
d) "If pigs are fish, then pigs are not fish." This is true, because "pigs are fish" is false.

Problem 5

a) Let P mean "I need to go to Oxnard" and Q mean "I need to go to Lampoc". We get $P \bigwedge Q$.
b) Let P mean "k is even," Q mean "k is bigger than 2," and R mean "k is prime". Then we get $(P \bigwedge Q) \to \sim R$.
c) Let P mean "It rains in the next week," Q mean "We will have vegetables," and R mean "We will have flowers". Then we get

$$[\sim P \to \sim (Q \bigvee R)] \bigwedge (P \to R)$$

This simplifies to $(Q \to P) \bigwedge (P \leftrightarrow R)$

Challenge Problem 1

This problem remains unsolved... Someone please solve it so that I can give everyone the solution!

Challenge Problem 2

The problem consisted of proving that there are an infinite number of primes of the form $6n + 1$. Recall that we originally proved that there were an infinite number of primes by doing the following:

Assume there is a finite number of primes $p_1, p_2, ..., p_n$. Now consider the number $x = p_1 p_2 ... p_n + 1$. This number is not divisible by any of the primes p_i, so it must be a new prime. This contradicts the assumption that the only primes were $p_1, p_2, ..., p_n$. Therefore there are an infinite number of primes.

Now remember that 2 and 3 are prime numbers. Therefore in the above proof, we can assume that 2 and 3 are in the set of p_i's – let's say $p_1 = 2$ and $p_2 = 3$. Then the new number x is of the form $x = 2 \cdot 3 \cdot p_3 p_4 ... p_n + 1 = 6k + 1$. And we know that x is prime – so x is a prime of the form $6n + 1$. Since there are an infinite number of primes, we can form an infinite number of "new primes" in the same way; hence there are an infinite number of primes of the form $6n + 1$.