Assignment 3 Solution

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Section 1.3, Exercise 1

(a) is not a closure operator because of the following counter-example: let $A = \{0\}$. Then $\mathbf{K}A = (0; \infty)$ which does not contain 0. This violates axiom C1.

(b) is a closure operator:

- C1 holds because for every element $a \in A$, we have $a \geq a$, so $a \in \mathbf{K}A$. Hence $A \subset KA$.

- C2: To show $\mathbf{K}A \bigcup \mathbf{K}B \subset \mathbf{K}(A \bigcup B)$, suppose $x \in \mathbf{K}A \bigcup \mathbf{K}B$.
  $\Rightarrow \exists a \in A : x \geq a$ OR $\exists a \in B : x \geq a$. We can assume without loss of generality (WLOG) that a is in A.
  $\Rightarrow \exists a \in A \bigcup B : x \geq a$.
  $\Rightarrow x \in \mathbf{K}(A \bigcup B)$
  $\Rightarrow \mathbf{K}A \bigcup \mathbf{K}B \subset \mathbf{K}(A \bigcup B)$.

  For the other direction, let x be an element of $\mathbf{K}(A \bigcup B)$.
  $\Rightarrow \exists a \in A : x \geq a$ OR $\exists a \in B : x \geq a$. We assume $a \in A$ WLOG.
  $\Rightarrow \exists a \in (A \bigcup B) : x \geq a$
  $\Rightarrow a \in \mathbf{K}(A \bigcup B)$
  $\Rightarrow \mathbf{K}(A \bigcup B) \subset \mathbf{K}A \bigcup \mathbf{K}B$.

- C3: We know that $\mathbf{K}A \subset \mathbf{KK}A$ from C1.
  Now let x be in $\mathbf{KK}A$.
  $\Rightarrow \exists a \in \mathbf{K}A : x \geq a$
  $\Rightarrow \exists b \in A : x \geq a \geq b$
  $\Rightarrow \exists b \in A : x \geq b$
  $\Rightarrow x \in \mathbf{K}A$
  $\Rightarrow \mathbf{KK}A \subset \mathbf{K}A$.

- C4 holds, because if I started out with nothing, then I can't take anything to be greater than that nothing, so the closure of the empty set is the empty set.

(c) is a closure operator:

- C1 holds because for any $a \in A$ I can write $a = 1 \cdot a$ and thus $a \in \mathbf{K}A$.
- C2: If x is in $\mathbf{K}(A \bigcup B)$, then x is a multiple of an element of $A \bigcup B$, so x is a multiple of an element of A or an element of B. Then x is in $\mathbf{K}A$ or x is in $\mathbf{K}B \rightarrow x \in \mathbf{K}A \bigcup \mathbf{K}B$.
  In the other direction, if x is in $\mathbf{K}A \bigcup \mathbf{K}B$, then x is a multiple of an element of A or x is a multiple of an element of B. So x is a multiple of an element of $A \bigcup B \rightarrow x \in \mathbf{K}(A \bigcup B)$
- C3: We know that $\mathbf{K}A \subset \mathbf{KK}A$ from C1.
  Let x be an element of $\mathbf{KK}A$.
  $\Rightarrow \exists a \in \mathbf{K}A : ka = x$ for some integer k.
  $\Rightarrow \exists b \in A : kk'b = x$ for some integers k and k'.
  $\Rightarrow x \in \mathbf{K}A$
  $\Rightarrow \mathbf{KK}A \in \mathbf{K}A$
- C4 holds because if I start out with nothing, then I don't have anything to multiply by a constant, so I can't get anything in return. So the closure of the empty set is the empty set.

(d) is not a closure operator because of the following counter-example:
Suppose $A = \{0\}$ and $B = \{2\}$. Then $\mathbf{K}A = \{0\}$ and $\mathbf{K}B = \{2\}$. So $\mathbf{K}A \bigcup \mathbf{K}B = \{0, 2\}$.
But $A \bigcup B = \{0, 2\}$ so $\mathbf{K}(A \bigcup B) = \{0, 1, 2\}$.
This violates axiom C2.

Section 1.4, Exercise 3

(a) We first show that any point in X is contained in at least one of the sets $IntA$, $ExtA$, and ∂A. To do this we simply need to show that $X = IntA \bigcup ExtA \bigcup \partial A$:

$$IntA \bigcup ExtA \bigcup \partial A = (X - \mathbf{K}(X-A)) \bigcup (X - \mathbf{K}A) \bigcup (\mathbf{K}A \bigcap \mathbf{K}(X-A))$$
$$= (\mathbf{K}(X-A))^c \bigcup (\mathbf{K}A)^c \bigcup (\mathbf{K}A \bigcap \mathbf{K}(X-A))$$
$$= [(\mathbf{K}(X-A))^c \bigcup (\mathbf{K}A)^c \bigcup \mathbf{K}A] \bigcap [(\mathbf{K}(X-A))^c \bigcup (\mathbf{K}A)^c \bigcup \mathbf{K}(X-A)]$$

Notice that $(\mathbf{K}A)^c \bigcup \mathbf{K}A = X$ and $(\mathbf{K}(X-A))^c \bigcup \mathbf{K}(X-A) = X$

So we get $IntA \bigcup ExtA \bigcup \partial A = X \bigcap X = X$

Now to show that a point cannot be in more than one of these sets, we need to show that their intersection is always empty:

$$IntA \bigcap ExtA = (X - \mathbf{K}(X-A)) \bigcap (X - \mathbf{K}A)$$
$$= (\mathbf{K}(A^c))^c \bigcap (\mathbf{K}A)^c$$

Now if a point is in $(\mathbf{K}(A^c))^c$, then it can't be in $\mathbf{K}(A^c)$, so it can't be in A^c, and it must be in A. So it must be in $\mathbf{K}A$ and it can't be in $(\mathbf{K}A)^c$ and hence the intersection here must be the empty set.

$$IntA \bigcap \partial A = (X - \mathbf{K}(X-A) \bigcap (\mathbf{K}A \bigcap \mathbf{K}(X-A))$$
$$= (\mathbf{K}(X-A))^c \bigcap \mathbf{K}A \bigcap \mathbf{K}(X-A)$$
$$= \emptyset \bigcap \mathbf{K}A = \emptyset$$

$$ExtA \bigcap \partial A = (X - \mathbf{K}A) \bigcap \mathbf{K}A \bigcap \mathbf{K}(X-A)$$
$$= (\mathbf{K}A)^c \bigcap \mathbf{K}A \bigcap \mathbf{K}(X-A)$$
$$= \emptyset \bigcap \mathbf{K}(X-A) = \emptyset$$

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The union of $IntA$, $ExtA$, and ∂A is the entire set X, and the intersection of any two of them is empty, so any point of X must be in exactly one of $IntA$, $ExtA$, and ∂A.

(b) $A \subset \mathbf{K}A$ and $(\mathbf{K}A \bigcap \mathbf{K}(X - A)) \subset \mathbf{K}A$, so $A \bigcup \partial A \subset \mathbf{K}A$.

For the other direction, let x be an element of $\mathbf{K}A$. Then x is not in $ExtA$ (definition of $ExtA$).
$\Rightarrow x \in (\partial A \bigcup IntA)$. (From part (a).)
Now notice that $x \in IntA$ implies x not in $\mathbf{K}(X - A)$.
$\Rightarrow x$ not in $X - A$, hence x is in A.
This means that $IntA \subset A$, so $x \in (\partial A \bigcup IntA)$ implies $x \in (\partial A \bigcup A)$.
$\Rightarrow \mathbf{K}A \subset (A \bigcup \partial A)$.

(c)We have just shown that $IntA \subset A$. From part (a), we know that $IntA \bigcap \partial A = \emptyset$. So we have $IntA \subset (A - \partial A)$.
For the other direction, let x be in $A - \partial A$.
$\Rightarrow x \in A \Rightarrow x \in \mathbf{K}A$
$\Rightarrow x$ is not in $ExtA$.
Furthermore, x is not in ∂A either, so from (a), x must be in $IntA$.
$\Rightarrow A - \partial A \subset IntA$

Section 1.4, Problem 4

(a) We know from the last problem that $IntA = A - \partial A$. It follows that $IntA \subset A$.

(b) $Int(A \bigcap B) = X - \mathbf{K}(X - (A \bigcap B)) = X - \mathbf{K}(A^c \bigcup B^c)$

$$= X - (\mathbf{K}(A^c) \bigcup \mathbf{K}(B^c))$$

$$= (X - \mathbf{K}(A^c)) \bigcap (X - \mathbf{K}(B^c))$$

$$= (X - \mathbf{K}(X - A)) \bigcap (X - \mathbf{K}(X - B))$$

$$= IntA \bigcap IntB$$

(c) $IntIntA = [X - \mathbf{K}(X - (X - \mathbf{K}(X - A)))] = [\mathbf{K}(((\mathbf{K}(A^c))^c)^c)]^c$

$$= [\mathbf{K}(\mathbf{K}(A^c))]^c$$

$$= [\mathbf{K}(A^c)]^c$$

$$= X - \mathbf{K}(X - A)$$

$$= IntA$$

(d) $IntX = X - \mathbf{K}(X - X) = X - K\emptyset = x - \emptyset = X$