Math S–101. Worksheet 2.
The Integers (I)

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Mathematical Induction and Well-Ordering

- **First Principle of Mathematical Induction.** Let $S(n)$ be a statement about integers for $n \in \mathbb{N}$ and suppose $S(n_0)$ is true for some integer n_0. If for all integers k with $k \geq n_0$ $S(k)$ implies that $S(k+1)$ is true, then $S(n)$ is true for all integers n greater than n_0.
- **Second Principle of Mathematical Induction.** Let $S(n)$ be a statement about integers for $n \in \mathbb{N}$ and suppose $S(n_0)$ is true for some integer n_0. If $S(n_0), S(n_0+1), \ldots, S(k)$ imply that $S(k+1)$ for $k \geq n_0$, then the statement $S(n)$ is true for all integers n greater than n_0.
- **Principle of Well-Ordering.** A nonempty subset S of $\mathbb{Z}$ is *well-ordered* if S contains a least element. Notice that the set $\mathbb{Z}$ is not well-ordered since it does not contain a smallest element. However, the natural numbers are well-ordered.

Problems

1. Prove that
$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$
for $n \in \mathbb{N}$.

2. Prove that $n! > 2^n$ for $n \geq 4$.

3. Prove that

$$x + 4x + 7x + \cdots + (3n - 2)x = \frac{n(3n - 1)x}{2}$$

for $n \in \mathbb{N}$.

4. Prove that $10^{n+1} + 10^n + 1$ is divisible by 3 for $n \in \mathbb{N}$.

5. Use induction to prove that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for $n \in \mathbb{N}$.

6. If x is a nonnegative real number, then show that $(1 + x)^n - 1 \geq nx$ for $n = 0, 1, 2, \ldots$.

7. Show that the Principle of Well-Ordering for the natural numbers implies that 1 is the smallest natural number. Use this result to show that the Principle of Well-Ordering implies the Principle of Mathematical Induction; that is, show that if $S \subset \mathbb{N}$ such that $1 \in S$ and $n + 1 \in S$ whenever $n \in S$, then $S = \mathbb{N}$.