Math S–101. Worksheet 9.
The Euclidean Closure Operator

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Distance

Given a set X, we say that a function $d : X \times X \to \mathbb{R}$ is a metric on X if

1. $d(x, y) \geq 0$ with equality if and only if $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

A set X together with a metric d is called a *metric space*. The *Euclidean metric* on $\mathbb{R}^n$ is defined to be

$$d_e(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

The Euclidean Closure Operator

For each $A \subset \mathbb{R}^n$, define the *Euclidean closure operator* to be the set $\mathbf{K}_e(A)$ of all $x \in \mathbb{R}^n$ such that for all $\epsilon > 0$ there exists an $a \in A$ with $d(x, a) < \epsilon$.

Theorem 1 *Let (X, d) be a metric space. Then*

$$\mathbf{K}(A) = \{x \in \mathbb{R}^n | \text{ for all } \epsilon > 0 \text{ there exists an } a \in A \text{ with } d(a, x) \leq \epsilon\}$$

defines a closure operator on X.

If $x \in \mathbb{R}^n$ and $r > 0$, we define the *open ball* of radius r centered at x to be

$$B(x; r) = \{y \in \mathbb{R}^n | d(x, y) < r\}.$$

Problems

1. If $a \in \mathbb{R}^n$, then $\mathbf{K}(\{a\}) = \{a\}$.
2. if $a < b$, then $\mathbf{K}(a, b) = [a, b]$.
3. if $a < b$, then $\mathbf{K}[a, b] = [a, b]$.
4. If $A = \{1, 1/2, 1/3, 1/4, \ldots\}$, then $\mathbf{K}(A) = A \cup \{0\}$.