Math S–101. Worksheet 13.
Limits and Continuity in $\mathbb{R}^n$

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Continuity at a Point

Let $(X, \mathbf{K}_X)$ and $(Y, \mathbf{K}_Y)$ be topological spaces. A function $f : X \to Y$ is *continuous* at $x \in X$ if for every $A \subset X$ with $x \in \mathbf{K}_X(A)$, $f(x) \in \mathbf{K}_Y(f(A))$.

Limits

Let $(X, \mathbf{K}_X)$ and $(Y, \mathbf{K}_Y)$ be topological spaces. Fix $a \in X$ and suppose that $f : (X \setminus \{a\}) \to Y$ is given. We write $\lim_{x \to a} f(x) = b$ and say that the *limit* of f as x approaches a exists if the function $g : X \to Y$ defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \neq a \\ b & \text{if } x = a \end{cases}$$

is continuous at a.

Epsilon-Delta Continuity

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is continuous if and only if for each $x \in X$ and each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$f(B(x; \delta)) \subset B(f(x); \epsilon).$$

Restating this result in terms of the Euclidean metric d, a function $f : X \to Y$ is continuous if and only if for each $x \in X$ and each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$d(x, y) < \delta \Longrightarrow d(f(x), f(y)) < \epsilon.$$

Making New Continuous Functions Out of Old Ones

If f and g are continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$ and $\alpha \in \mathbb{R}$, then

1. $\alpha \cdot f$ is continuous.
2. $f + g$ is continuous.
3. $f \cdot g$ is continuous.
4. f/g is continuous provided that $g(x) \neq 0$

Some Problems

1. Use the $\epsilon{-}\delta$ notion of continuity to show that the function $f(x) = 3x{-}2$ is continuous.
2. Use the $\epsilon - \delta$ notion of continuity to show that the function $f(x) = x^2$ is continuous.