Math S–101. Worksheet 16.
The Brouwer Fixed Point Theorem

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The Fixed Point Property

A space $(X, \mathbf{K})$ has the fixed point property if every continuous function $f : X \to X$ has a fixed point.

Theorem 1 *The fixed point property is a topological invariant.*

Retractions[1]

Let $(X, \mathbf{K})$ be a topological space and suppose that $A \subset X$. A *retraction* from X to A is a continuous function $r : X \to A$ such that $r(a) = a$ for all $a \in A$.

Lemma 2 (Retraction Lemma) *If there is a continuous function $f : B^2 \to B^2$ with no fixed points, then there exists a retraction from B^2 to S^1, the unit circle.*

Lemma 3 *There is a homeomorphism $h : B^2 \to I^2$ such that*

$$x \in S^1 \Longleftrightarrow h(x) \in \partial I^2.$$

Lemma 4 (Retraction Lemma 2) *To prove the Brower Fixed Point Theorem, it is enough to show that there is no retraction from I^2 to ∂I^2.*

[1] We define the subsets S^1 and I^2 of $\mathbb{R}^2$ to be $S^1 = \{x \in \mathbb{R}^2 | d(0, x) = 1\}$ and $I^2 = [-1, 1] \times [-1, 1]$.

Uniform Continuity

Let X and Y be subsets of Euclidean spaces. A function $f : X \to Y$ is *uniformly continuous* if for all $\epsilon > 0$, there exists a $\delta > 0$ such that $d(f(x), f(y)) < \epsilon$ whenever $d(x, y) < 0$.

Remarks. Uniform continuity is a much stronger notion than continuity. To prove continuity, you are given an $x \in X$ and $\epsilon > 0$ and you need to find a $\delta > 0$. To prove uniform continuity, you are given an $\epsilon > 0$ and you need to find a *single* $\delta > 0$ that works for *every* $x \in X$. Uniform continuity only makes sense in metric spaces.

Theorem 5 (Uniform Continuity Lemma) *If $f : I^2 \to \mathbb{R}^n$ is continuous, then f is uniformly continuous.*

The Game of Hex

Theorem 6 *If every hexagon of a Hex board is colored black or white then one of the two players has a winning chain.*

The Brouwer Fixed Point Theorem[2]

Theorem 7 *Every continuous map $f : B^2 \to B^2$ has a fixed point.*

Step 1. The fixed point property is a topological invariant (Theorem 1). Since B^2 and I^2 are homeomorphic by Theorem 3.

Step 2. To prove the Brower Fixed Point Theorem, it is enough to show that there is no retraction from I^2 to ∂I^2 (Lemma 4). Suppose there is a retraction $r : I^2 \to \partial I^2$. We will show that this leads to a contradiction.

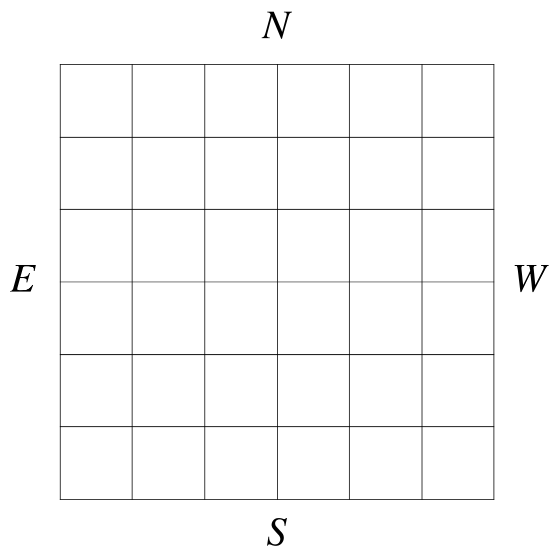
Step 3. By the *Uniform Continuity Theorem* (Theorem 5), we can choose $\delta > 0$ such that

$$|x - y| < \delta \Longrightarrow |r(x) - r(y)| < \epsilon = 2.$$

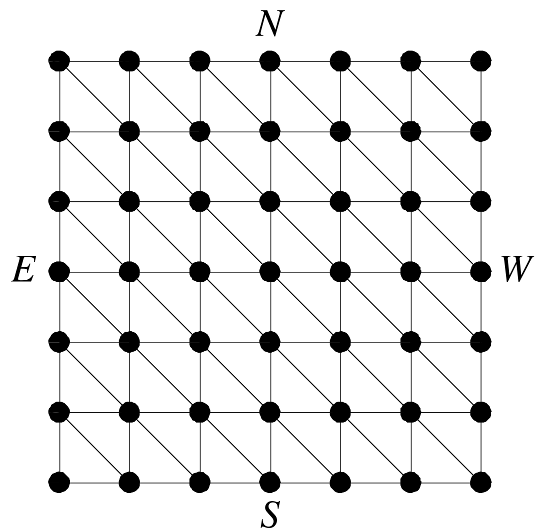
[2]The proofs of Theorems 1–6 can be found in Prof. Goroff's notes.

This means that if $x, y \in I^2$ with $d(x, y) < \delta$, then $d(r(x), r(y)) < 2$. In particular, no two points in I that are less than δ apart can be mapped to opposite sides of the square, ∂I^2.

Step 4. Choose a positive integer n such that $2\sqrt{2}/n$ and divide I^2 into n^2 squares, each with side $2/n$. Since the diagonal of each of these small squares is $2\sqrt{2}/n$, no two points in the same small square are mapped by r to opposite sides of ∂I^2.

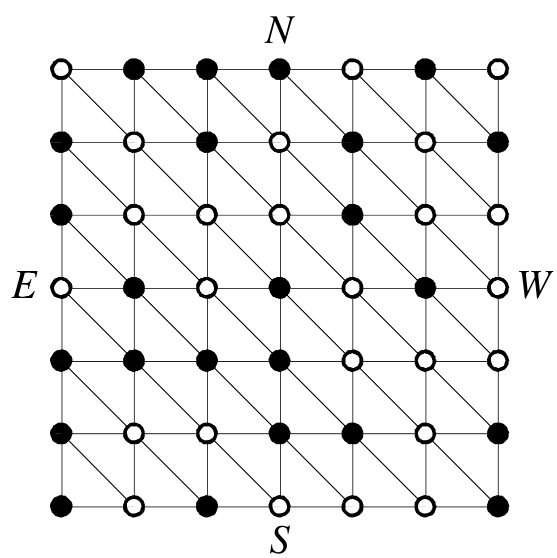


Step 5. To make I^2 equivalent to a hex board, add diagonal lines slanting downward left to right. The corners of the small squares (the bold discs) correspond to hexagons. Notice that each corner has six rays edges extending outward. These edges correspond to the edges of our hexagon. Two hexagons are adjacent if and only if there is a single line segment connecting the corresponding points.



Step 6. Number each hexagon (point) by N, S, E, or W, depending on which side the corresponding hexagon (point) is mapped by r. If a hexagon (point) is mapped to two adjacent sides of ∂I^2 (i.e., a corner of ∂I^2), then label it either way.

Step 7. Color a hexagon (point) white if it is labeled N or S and black if it is labeled E or W.



Step 8. By the Hex Lemma (Theorem 6), one of the players has a winning chain. Without loss of generality, we can assume that three is a chain of white hexagons connecting the top and bottom (N and S). Since r is a retraction, any hexagon (point) in the chain near the top is labeled N and any hexagon (point) near the bottom is labeled S. This means that somewhere in the middle of the chain, there must be a hexagon (point) labeled as N that is adjacent to a hexagon (point) labeled S. Both of these hexagons can be thought of as points on the same square. However, points on the same square cannot be mapped to opposite sides of ∂I^2 by Step 4. This is our contradiction. Therefore, r cannot be a retraction.