MATH 112 SET 8 SOLUTIONS

ALEX WALDRON

Feel free to email waldron@fas.harvard.edu if anything is unclear. Also, if you think I misunderstood any of your arguments or was too harsh, just email me and we can meet to chat about it/negotiate (this applies for any CA).

6.4) Choose any partition P of $[a, b]$. Since $\mathbb{Q}$ and $\mathbb{Q}^c$ are dense, $M_i = 1$ and $m_i = 0$ $\forall\, i$. Thus $\sum(M_i - m_i)\Delta x_i = \sum \Delta x_i = |b - a|$ so $f \notin \mathcal{R}$.

6.5 No to the first question: take f from the previous problem, and consider $g := 2f - 1$, not integrable else $f = (g+1)/2$ would be. Then $g^2(x) = 1$ on all of $[a, b]$, integrable.

However, f is integrable if f^3 is, for a any real number has a unique real cube root: so $f = \sqrt[3]{f^3}$ which is a composition of a continuous with an integrable function, hence integrable.

6.7 (a) Given $f \in \mathcal{R}$ on $[0, 1]$, we need $\int_0^1 f(x)dx = \lim_{c\to 0} \int_c^1 f(x)dx$.

Use $|\int_0^1 f(x)\, dx - \int_c^1 f(x)\, dx| = |\int_0^c f(x)\, dx| \leq (c-0)M$ where $|f| \leq M$ on $[0, 1]$, which is the desired expression since $c > 0$ is arbitrary.

(b) Note that from (a), we can show that the value of any (regular) Riemann integral of f on $[a, b]$ does not depend on $f(a)$ and $f(b)$.

Let $n(x) = \lfloor \frac{1}{x} \rfloor$. Now define

$$\begin{aligned} f &: (0,1] \Rightarrow \mathbb{R} \\ f(x) &= (-1)^{n(x)} n(x) \end{aligned}$$

f is constant on the intervals $(1/(n+1), 1/n]$. On any interval $[c, 1]$, $c > 0$, f is bounded with finitely many discontinuities, hence integrable. So we can split up the domain into finitely many intervals,

$$I(c) := \int_c^1 f(x)\, dx = \int_c^{1/n(c)} f(x)\, dx + \sum_{i=1}^{n(c)-1} \int_{\frac{1}{i+1}}^{\frac{1}{i}} f(x)\, dx.$$

Now compute $\int_{\frac{1}{i+1}}^{\frac{1}{i}} f(x)dx$. Choose any P on $[1/(i+1), 1/i]$. Then since $f(x) = i$, constant on all but one endpoint, this sub-integral has value

$$i(-1)^i(\frac{1}{i} - \frac{1}{i+1}) = i(-1)^i(\frac{1}{i(i+1)}) = (-1)^i \frac{1}{i+1}.$$

Likewise, $\int_c^{1/n(c)} f(x)dx = (-1)^{n(c)}(1 - n(c)/c)$. Therefore,

$$\lim_{c\to 0} I(c) = \lim_{c\to 0} \left((-1)^i(1 - n(c)/c) + \sum_{i=1}^{n(c)-1} \frac{1}{i+1}(-1)^i \right) = 0 + \sum_{i=1}^{\infty} (-1)^i \frac{1}{i+1}$$

which converges.

This holds with $|f|$ except that $(-1)^i$ is omitted throughout. Then $\int_c^1 |f(x)|dx = \sum_{i=1}^{n(c)-1} \frac{1}{i+1}$, the harmonic series, which diverges as $c \to 0$.

6.10) (a) This is equivalent to $p/q = p - 1$ and $q/p = q - 1$. First we will show that if $u^p = v^q$ then equality holds.

$$u^p/p + v^q/q = u^p(1/p + 1/q) = u^p = uu^{p-1} = u(u^p)^{1/q} = uv.$$

We will show strict inequality for an arbitrary $u, v \geq 0$ such that $u^p \neq v^q$; assume wlog $u^p < v^q$. Let $f(x) = u^p/p + x^q/q - ux$. Then we claim that $f(v) > 0$. If this holds, then we will have shown equality when $u^p = v^q$ and strict inequality otherwise, which is the claim.

Choose v' such that $u^p = v'^q$, then $v'^q < v^q \Rightarrow v' < v$ since $p, q > 0$ and $u, v \geq 0$. Then from the mean value theorem, there is some $x \in (v', v)$ such that $(v - v')f'(x) = f(v) - f(v') = f(v)$. So we'll be done if $f'(x) > 0$ on (v', v). Then for $v' < x < v \Rightarrow u^p < x^q$, have $f'(x) = x^{q-1} - u = (x^q)^{1/p} - u > u^{p/p} - u = 0$.

(b) We have the inequality in (a) for f, g. Then it is preserved under an integral: $\int_a^b fg\,dx \leq \int_a^b f^p\,dx/p + \int_a^b g^q\,dx/q = 1/p + 1/q = 1$.

(c) Let $\tilde{f} = \frac{|f|}{(\int_a^b |f|^p\,dx)^{1/p}}$, and $\tilde{g}$ likewise. Then $\int_a^b \tilde{f}^p\,dx = 1 = \int_a^b \tilde{g}^q\,dx$ so the inequality from (b) holds for $\tilde{f}, \tilde{g}$. But we can write

$$\begin{aligned}
1 = \left\{\int_a^b \tilde{f}^p\,dx\right\}^{\frac{1}{p}} \left\{\int_a^b \tilde{g}^q\,dx\right\}^{\frac{1}{q}} &\geq \int_a^b \tilde{f}\tilde{g}\,dx \\
\frac{\left\{\int_a^b |f|^p\,dx\right\}^{\frac{1}{p}} \left\{\int_a^b |g|^q\,dx\right\}^{\frac{1}{q}}}{(\int_a^b |f|^p\,dx)^{\frac{1}{p}}(\int_a^b |g|^q\,dx)^{\frac{1}{q}}} &\geq \frac{\int_a^b |f||g|\,dx}{(\int_a^b |f|^p\,dx)^{\frac{1}{p}}(\int_a^b |g|^q\,dx)^{\frac{1}{q}}} \\
\left\{\int_a^b |f|^p\,dx\right\}^{\frac{1}{p}} \left\{\int_a^b |g|^q\,dx\right\}^{\frac{1}{q}} &\geq |\int_a^b fg\,dx|
\end{aligned}$$

(d) We have, by taking the limit of each side and from 4.4c,

$$\begin{aligned}
|\lim_{c\to 0} \int_c^1 fg\,dx| &\leq \lim_{c\to 0} \left\{\int_c^1 |f|^p\,dx\right\}^{\frac{1}{p}} \left\{\int_c^1 |g|^q\,dx\right\}^{\frac{1}{q}} \\
&= \left\{\lim_{c\to 0}\int_c^1 |f|^p\,dx\right\}^{\frac{1}{p}} \left\{\lim_{c\to 0}\int_c^1 |g|^q\,dx\right\}^{\frac{1}{q}}
\end{aligned}$$

This is the claim.

6.11) Let $a = f - g$, $b = f - h$, $c = h - g$, then $a = b + c$. Then

$$\begin{aligned}
(\|a\|_2)^2 &= \int_a^b |b + c|^2\,dx \leq \int_a^b (|b|^2 + |c|^2 + 2|b||c|)\,dx \\
&\leq \int_a^b |b|^2\,dx + \int_a^b |c|^2\,dx + 2\int_a^b |b||c|\,dx \\
&\leq \int_a^b |b|^2\,dx + \int_a^b |c|^2\,dx + 2\left(\int_a^b |b|^2\,dx \int_a^b |c|^2\,dx\right)^{1/2} = (\|b\|_2 + \|c\|_2)^2.
\end{aligned}$$

6.12) Choose a partition P such that $U(P, f) - L(P, f) < \epsilon$, and construct g from the hint corresponding to this partition, checking that the definition makes sense at each x_i. The function g is a linear polynomial on each (x_i, x_{i+1}) and thus the left and right limit of g at x_i are $f(x_i) = g(x_i)$, so g is continuous on all of $[a, b]$. Since g is linear on (x_i, x_{i+1}), $m_i \leq f(x_i) \leq g(x) \leq f(x_{i\pm1}) \leq M_i$ (whichever is greater), so $|g(x) - f(x)| \leq M_i - m_i \, \forall \, x \in [x_i, x_{i+1}]$. Since f is integrable, $|f(x) - f(x')| \leq B$ on $[a, b]$ for some B; thus

$$0 \leq (\|g - f\|_2)^2 \leq U(P, |g - f|^2) \leq \sum (M_i - m_i)^2 \Delta x_i \leq \sum B(M_i - m_i)\Delta x_i \leq B\epsilon.$$

This suffices since ϵ was arbitrary.