Problem Set # 9 Solutions

Igor Rapinchuk

1. (Ch. 7, # 2) Suppose $f_n(x) \longrightarrow f(x)$ and $g_n(x) \longrightarrow g(x)$ uniformly on E. We claim that $f_n(x) + g_n(x) \longrightarrow f(x) + g(x)$ uniformly on E. To show this, fix $\varepsilon > 0$. Then $f_n \to f$ and $g_n \to g$ uniformly, there exist N_1, N_2 such that

$$|f_n(x) - f(x)| < \varepsilon/2 \text{ for all } n \geqslant N_1$$

and

$$|g_n(x) - g(x)| < \varepsilon/2 \text{ for all } n \geqslant N_2$$

and ALL $x \in E$. Then for $n \geqslant N := \max(N_1, N_2)$ we have

$$|(f_n(x) + g_n(x)) - (f(x) + g(x))| \leqslant |f_n(x) - f(x)| + |g_n(x) - g(x)| <$$
$$\varepsilon/2 + \varepsilon/2 = \varepsilon$$

for all $x \in E$, proving our claim.

Now make the additional assumption that $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions. Then, by exercise #1 on page 165, these sequences are *uniformly* bounded. So, there exist M_1, M_2 such that $|f_n(x)| \leq M_1, |g_n(x)| \leq M_2$ for all $x \in E$ and all n. Let $M = \max(M_1, M_2)$. Now let $\varepsilon > 0$. Then, there exists N_1 such that for all $n, m \geq N_1$,

$$|f_n(x) - f_m(x)| < \frac{\varepsilon}{2M}, \forall x \in E$$

and there exists N_2 such that for all $n, m \geq N_2$,

$$|g_n(x) - g_m(x)| < \frac{\varepsilon}{2M}, \forall x \in E.$$

Let $N = \max(N_1, N_2)$. Then for all $n \geq N$, we have

$$|f_n g_n - f_m g_m| = |f_n g_n - f_n g_m + f_n g_m - f_m g_m| \leq |f_n g_n - f_n g_m| + |f_n g_m - f_m g_m| \leq$$
$$|f_n||g_n - g_m| + |g_m||f_n - f_m| < M \cdot \frac{\varepsilon}{2M} + M \cdot \frac{\varepsilon}{2M} = \varepsilon, \forall x \in E.$$

So, $\{f_n g_n\}$ converges uniformly on E.

2. (Ch. 7, # 3) Let $f_n(x) = g_n(x) = x + (1/n)$ on $E = \mathbb{R}$. Then the sequence $f_n(x) = g_n(x)$ uniformly converges to $f(x) = g(x) = x$. Indeed,

$$|f_n(x) - f(x)| = 1/n < \varepsilon$$

for all $n \geqslant N := [1/\varepsilon] + 1$ and all $x \in E$. Then $f_n(x)g_n(x) \longrightarrow x^2$ pointwise on E. However, the convergence is not uniform because

$$|f_n(x)^2 - f(x)^2| = |(2/n)x + (1/n^2)|$$

in particular

$$|f_n(n)^2 - f(n)^2| > 2$$

This means that, for example, for $\varepsilon = 1$ it is not possible to find N such that for $n \geqslant N$ we have

$$|f_n(x)^2 - f(x)^2| < \varepsilon \text{ for all } x \in E,$$

so there is no uniform convergence.

3. (Ch. 7, # 4) Let $f_n(x) = \dfrac{1}{1+n^2x}$. Then for any $x \neq 0$ we have

$$\frac{|f_n(x)|}{1/n^2} = \frac{n^2}{|1+n^2x|} = \frac{1}{|1/n^2+x|} \longrightarrow \frac{1}{|x|} \quad \text{as} \quad n \to \infty.$$

So, by the Limit Comparison Test, since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the series $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely. Of course, for $x = 0$, the series $\sum_{n=1}^{\infty} f_n(0) = \sum_{n=1}^{\infty} 1$ diverges. Notice that $f_n(-1/n^2)$ is undefined, but still the series $\sum_{k=1,k\neq n}^{\infty} f_k(-1/n^2)$ converges absolutely. Thus, f is defined on $E = \mathbb{R}\setminus\Gamma$ where $\Gamma = \{0\}\cup\{-1/n^2 \mid n = 1, 2, \ldots\}$.

If $|x| \geqslant \alpha > 0$ then

$$|1+n^2x| \geqslant n^2\alpha - 1 \geqslant n^2\alpha/2$$

for $n \geqslant n_\alpha = [\sqrt{2/\alpha}] + 1$, implying that

$$\left|\frac{1}{1+n^2x}\right| \leqslant \frac{2}{n^2\alpha} \tag{1}$$

for $n \geqslant n_\alpha$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2\alpha}$ converges, by the Weierstrass M-test (Theorem 7.10), the series $\sum_{n=n_\alpha}^{\infty} f_n(x)$ converges uniformly on $\{x \mid |x| > \alpha\}$. On the other hand, the finite sum $\sum_{n=1}^{n_\alpha-1} f_n(x)$ is defined everywhere on $\mathbb{R}\setminus\Gamma$. So, we obtain that the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $E_\alpha = \{x \in \mathbb{R}\setminus\Gamma \mid |x| > \alpha\}$, for any $\alpha > 0$.

We claim that the series does not converge uniformly already on $(0, +\infty)$, and therefore also on $\mathbb{R}\setminus\Gamma$. Indeed, if $F_n(x) = \sum_{n=1}^{\infty} f_n(x)$ and $F_n(x) \to f(x)$ uniformly on $(0, +\infty)$ then $f_n(x) = F_n(x) - F_{n-1}(x)$ must converge to 0 uniformly on $(0, +\infty)$. In our situation,

$$f_n(1/n^2) = 1/2$$

showing that $f_n \not\longrightarrow 0$ uniformly on $(0, +\infty)$, so the series does not converge uniformly.

Observe that every point $x_0 \in E$ is contained in E_α for, say, $\alpha = |x_0|/2$. We have seen that the series converges uniformly on E_α, implying that $f(x)$ is continuous on E_α (Theorem 7.12 applied to partial sums), and hence at x_0. Since $x_0 \in E$ is arbitrary, f is continuous on E.

Finally, let us show that f is not bounded already on $(0, +\infty)$. Indeed, let N be an arbitrary integer. Then for $n \leqslant N^2$ we have

$$f_n(1/N^2) = \frac{1}{1 + n^2/N^2} \geqslant 1/2$$

implying that $f(1/N^2) \geqslant N^2/2$, so f is not bounded on $(0, \infty)$. On the other hand, f is bounded on $(\alpha, +\infty)$ for any $\alpha > 0$. Indeed, for any $x \in (\alpha, +\infty)$ we have

$$0 < f(x) \leqslant \sum_{n=1}^{\infty} \frac{1}{n^2 x} < \frac{1}{\alpha} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$