Homework Assignment # 1

Due Thursday, September 20

Show all your work, and write up your solutions as neatly as possible. See the course syllabus for the cooperation policy on homeworks. You may use any formulas which are stated in chapters 1 and 2 of the book, even if we did not specifically discuss them in class. You may also use the identities from exercise 4 on page 17 if needed.

1. Solve the following equations in complex numbers:
   a. $z^2 + \sqrt{-32}z - 6i = 0$
   b. $z^8 + 1 = 0$

2. Assuming either that $|z| = 1$ or $|w| = 1$, and that $|\overline{z}w| \neq 1$, show that
$$|\frac{z-w}{1-\overline{z}w}| = 1.$$

3. Suppose $P \in \mathbf{C}[x, y]$ is a polynomial in both $x + iy$ and $x - iy$. Show that P is constant.

4. Let $P \in \mathbf{C}[z]$ be a nonconstant analytic polynomial. Show that $P(z) \to \infty$ as $z \to \infty$. [See Definition 1.11 on p.16 of the text.]

5. Show that z and z' in $\mathbf{C}$ correspond (via stereographic projection) to diametrically opposite points on the Riemann sphere Σ iff $\overline{z} = \frac{1}{z'}$.

6. Let $\theta \in \mathbf{R}$, and suppose θ is not a multiple of 2π. Let $\alpha = cos(\theta) - 1, \beta = sin(\theta), \alpha' = cos((n+1)\theta) - 1, \beta' = sin((n+1)\theta)$. Show that
$$1 + cos\theta + cos2\theta + \cdots + cosn\theta = \frac{\alpha\alpha' + \beta\beta'}{\alpha^2 + \beta^2}.$$

7. Suppose $\{a_n\}$ is a sequence of positive real numbers and
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L.$$
Show that $\lim_{n\to\infty} a_n^{1/n} = L$. Use this to find the radius of convergence of
$$\sum_{n=1}^{\infty} \frac{n!z^n}{n^n}.$$