Homework Assignment # 3

Due Thursday, October 11

1. a. Fix $y_0 \in \mathbf{R}$, and let $A_{y_0} = \{z \in \mathbf{C} \mid \mathrm{Im}(z) \in [y_0, y_0 + 2\pi)\}$. Show that $e^z : A_{y_0} \to \mathbf{C} - \{0\}$ is a bijection.

   b. Define the branch of the logarithm with range equal to A_{y_0} to be the function $\log_{y_0} : \mathbf{C} - \{0\} \to A_{y_0} \subseteq \mathbf{C}$ defined by $\log_{y_0}(z) = \log|z| + i\arg_{y_0}(z)$, where $\arg_{y_0}$ takes its values in $[y_0, y_0+2\pi)$. Show by example that $\log_{y_0}$ is not continuous on $\mathbf{C} - \{0\}$, and that we do not always have $\log_{y_0}(z_1 z_2) = \log_{y_0}(z_1) + \log_{y_0}(z_2)$.

   c. Show, however, that if $z_1, z_2 \in \mathbf{C} - \{0\}$, then $\log_{y_0}(z_1 z_2)$ and $\log_{y_0}(z_1) + \log_{y_0}(z_2)$ must differ by a multiple of $2\pi i$.

2. If $F : \mathbf{C} \to \mathbf{C}$ is an entire function and $F' \equiv 0$, show that F is a constant.

3. Evaluate the line integral $\int_C x\ dz$, where C is the boundary of the unit square $[0, 1] \times [0, i]$ with its standard (counterclockwise) orientation.

4. Evaluate the line integral $\int_C (z - i)\ dz$, where C is the parabolic segment

$$z(t) = t + it^2,\ -1 \leq t \leq 1$$

   a. Directly, using the definition of the line integral

   b. By applying Proposition 4.12 (the complex analogue of the fundamental theorem of calculus)

   c. By integrating directly along the straight line from $-1+i$ to $1+i$ and applying the closed curve theorem.

5. Use the ML-inequality to show that

$$\left| \int_C \frac{e^z}{z} dz \right| \leq \pi e,$$

where C is the upper half of the unit circle in $\mathbf{C}$, traversed counterclockwise.

6. Show that if f is a continuous function from **C** to the interval $[-M, M] \subset$ **R**, then

$$|\int_{S^1} f(z)dz| \leq 4M,$$

where S^1 is the unit circle in **C**, parametrized in the standard way. [You may wish to follow the outline of proof suggested on pp. 267–268 of the book, but be sure to fill in all of the details and justify each step.]