# Homework Assignment # 8

Due Thursday, December 6

1. Derive the fundamental theorem of algebra as a consequence of Rouché's theorem.

2. Find the number of zeros of

   a. $f(z) = e^z - 2z - 1$ in $|z| \leq 1$

   b. $f(z) = z^6 + 9z^4 + z^3 + 2z + 4$ in $|z| \leq 1$

   c. $f(z) = z^4 - 5z + 1$ in $1 \leq |z| \leq 2$

3. Suppose $f$ is entire and that $f(z)$ is real iff $z$ is real. Use the argument principle to show that $f$ can have at most one zero.

4. a. Let $k$ be an integer. Prove directly that $f(z) = z^k$ is locally injective for $z \neq 0$.

   b. Find the largest disk centered at $z = 1$ on which the function $f(z) = z^4$ is injective.

5. Near what points are the following maps conformal?

   a. $f(z) = \overline{z}$

   b. $f(z) = \tan(z)$

6. What is the image of a horizontal or vertical line in $\mathbf{C}$ under the mapping $f(z) = e^z$? Justify your answer.

7. The *Fourier transform* of a function $f : \mathbf{R} \to \mathbf{C}$, if it exists, is a function $\hat{f} : \mathbf{R} \to \mathbf{C}$ defined by the improper integral

   $$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx.$$

   The Fourier transform has many applications in mathematics and physics.

   The *normal distribution probability function* is given by the formula

   $$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Prove that $f$ is equal to its own Fourier transform, i.e., that the improper integral defining the Fourier transform of $f$ exists, and that $\hat{f}(\omega) = f(\omega)$ for all $\omega \in \mathbf{R}$. (You may use the standard fact from real analysis that $\int_{-\infty}^{\infty} e^{-x^2/2}dx = \sqrt{2\pi}$.) [**Hint:** Prove that $\int_{-\infty}^{\infty} e^{-(x+i\omega)^2/2}dx = \sqrt{2\pi}$ for all $\omega \in \mathbf{R}$ by considering the contour integral of $g(z) = e^{-z^2/2}$ around a rectangle with vertices at $-R, R, R+\omega i, -R+\omega i$ and letting $R \to \infty$.]