# Math 113, Fall 2001

## Solutions to Problem Set 1

September 26, 2001

**1a.** Using the quadratic formula,

$$\begin{aligned} z &= \frac{-4\sqrt{2}i \pm \sqrt{-32+24i}}{2} \\ &= -2\sqrt{2}i \pm \sqrt{-8+6i} \\ &= -2\sqrt{2}i \pm (1+3i) \\ &= 1+(3-2\sqrt{2})i, 1+(-3-2\sqrt{2})i. \end{aligned}$$

**1b.** We have $z^8 = -1 = e^{(2n+1)\pi i}$, so $z = e^{\frac{(2n+1)\pi i}{8}}$. Thus $z$ can be any of $e^{\frac{\pi i}{8}}$, $e^{\frac{3\pi i}{8}}$, $e^{\frac{5\pi i}{8}}$, $e^{\frac{7\pi i}{8}}$, $e^{\frac{9\pi i}{8}}$, $e^{\frac{11\pi i}{8}}$, $e^{\frac{13\pi i}{8}}$ or $e^{\frac{15\pi i}{8}}$.

**2.** If $|z| = 1$, then $|\overline{z}| = 1$, so

$$\begin{aligned} \left|\frac{z-w}{1-\overline{z}w}\right| &= \left|\frac{\overline{z}z-\overline{z}w}{1-\overline{z}w}\right| \\ &= \left|\frac{\overline{z}z-\overline{z}w}{\overline{z}z-\overline{z}w}\right| \\ &= 1. \end{aligned}$$

A similar technique shows that if $|w| = 1$, then the result holds.

**3.** If $P$ is a polynomial in $z$, then $P$ satisfies the Cauchy-Riemann equation $\frac{\partial P}{\partial x} = -i\frac{\partial P}{\partial y}$. But if $P$ is a polynomial in $\overline{z}$, one can show that $P$ satisfies the analogous equation $\frac{\partial P}{\partial x} = i\frac{\partial P}{\partial y}$. Hence $\frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = 0$ and $f$ is a constant.

**4.** Suppose $P = a_0 + a_1 z + \ldots + a_n z^n$ with $a_n \neq 0$. Some manipulation of the triangle inequality yields $|P(z)| \geq |a_n z^n| - |a_0 + a_1 z + \ldots + a_{n-1}z^{n-1}|$. If $M = \max\{a_0, a_1, \ldots, a_{n-1}\}$ and $|z| > 1$, then $|a_0 + a_1 z + \ldots a_{n-1}z^{n-1}| \leq M|z|^{n-1}$. Hence for $|z|$ large enough, $|P(z)| \geq |a_n||z|^n - M|z|^{n-1}$. Clearly the right-hand side approaches infinity as $|z|$ approaches infinity.

**6.** Note that $1 + \cos\theta + \ldots \cos n\theta = \mathrm{Re}(1 + e^{i\theta} + \ldots + e^{ni\theta})$. If $\theta$ is not a multiple of $2\pi$, then $e^{i\theta} \neq 1$, and the right-hand side equals $\frac{1-e^{(n+1)i\theta}}{1-e^{i\theta}}$. Now

$$\frac{1-e^{(n+1)i\theta}}{1-e^{i\theta}} = \frac{1-e^{(n+1)i\theta}}{1-e^{i\theta}}\frac{1-e^{-i\theta}}{1-e^{-i\theta}}$$

$$= \frac{1 - e^{-i\theta} - e^{(n+1)i\theta} + e^{ni\theta}}{2 - 2\cos\theta}.$$

The real part of this expression is $\frac{1-\cos\theta-\cos(n+1)\theta+\cos(n\theta)}{2-2\cos\theta}$, which is easily checked to be equal to $\frac{\alpha\alpha'+\beta\beta'}{\alpha^2+\beta^2}$.

**7.** For any $\epsilon > 0$, we know that there exists $N$ such that for $n \geq N$, $L - \epsilon < \frac{a_n+1}{a_n} < L + \epsilon$. Hence for $k > 0$, $a_{N+k}$ lies between $a_N(L-\epsilon)^k$ and $a_N(L+\epsilon)^k$, or between $a_N(L-\epsilon)^{-N}(L-\epsilon)^{N+k}$ and $a_N(L+\epsilon)^{-N}(L+\epsilon)^{N+k}$.

Now $a_N(L-\epsilon)^{-N}$ and $a_N(L+\epsilon)^N$ are both positive constants, so

$\lim_{k\to\infty}(a_N(L-\epsilon)^{-N})^{\frac{1}{N+k}} = \lim_{k\to\infty}(a_N(L+\epsilon)^{-N})^{\frac{1}{N+k}} = 1$. Hence there exists some $M > N$ such that for $m \geq M$, $(a_N(L-\epsilon)^{-N})^{\frac{1}{m}}$ and $(a_N(L+\epsilon)^{-N})^{\frac{1}{m}}$ are both between $1-\epsilon$ and $1+\epsilon$.

Putting the two pieces together, we can conclude that for $k$ large enough, $(a_{N+k})^{\frac{1}{N+k}}$ lies between $(1-\epsilon)(L-\epsilon)$ and $(1+\epsilon)(L+\epsilon)$. If we take $\epsilon < 1$, we can say that for $k$ large enough, $(a_{N+k})^{\frac{1}{N+k}}$ lies between $(L-3\epsilon)$ and $(L+3\epsilon)$. If $\lim_{n\to\infty}(a_n)^{\frac{1}{n}} \neq L$, then there exists $\epsilon > 0$ such that for infinitely many $n$, $a_n^{\frac{1}{n}}$ lies outside of the interval $(L-3\epsilon, L+3\epsilon)$, which cannot happen. Hence $\lim_{n\to\infty} a_n^{\frac{1}{n}} = L$.

The ratio of successive terms in the infinite series $\sum_{n=1}^{\infty} \frac{n!z^n}{n^n}$ is $z(\frac{n}{n+1})^n$, and this limit as $n \to \infty$ is $\frac{z}{e}$. Hence the radius of convergence of this series is $e$.