Solution Set 4

Fall 2001
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Math 113

Problem 1.

Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function, and let C be a smooth curve in $\mathbb{C}$ parametrized by $z(t) : [a, b] \to \mathbb{C}$. Let $P = (a_0 = a, a_1, a_2, \ldots, a_n = b)$ be a partition of $[a, b]$, let $z_j = z(a_j)$ for $j = 0 \ldots n$ and let $R_P(f) = \sum_{j=1}^{n} f(z_j)(z_j - z_{j-1})$. Explain why $\int_C f \approx R_P(f)$ if $\max_j |a_j - a_{j-1}|$ is small. Formulate and prove a more precise version of this statement.

We prove that for each $\epsilon > 0$ there exists a δ such that if $\max_j |a_j - a_{j-1}| < \delta$, $|\int_C f dz - R_P(f)| \le \epsilon$:

$$R_P(f) = \sum_{j=1}^{n} f(z(aj))(\int_{z(a_{j-1})}^{z(a_j)} dz)$$

where the integral is taken along the curve C, which we can do because the function 1 is the derivative of z and the fundamental theorem of calculus still holds in the complex case as shown in class. This implies

$$|\int_C f dz - R_P(f)| = |\sum_{j=1}^{n} \int_{z(a_{j-1})}^{z(a_j)} f(z) - f(z_j) dz|.$$

Let the length of C be L. Because f is continuous and C is compact, f is uniformly continuous. We therefore have a δ' such that for all $z_1, z_2 \in C$ with $|z_1 - z_2| < \delta'$, $|f(z_1) - f(z_2)| < \epsilon/L$. Because C is parameterized by the continuous function $z : [a, b] \to \mathbb{C}$ and $[a, b]$ is compact, z is also uniformly continuous. This means we can choose δ such that $|a_j - a_{j-1}| < \delta$ implies $|z(a_j) - z(a_{j-1})| < \delta'$. This δ is as in our claim because $max_j |a_j - a_{j-1}| < \delta$ implies

$$|\int_C f dz - R_P f| \le |\sum_{j=1}^{n} \int_{z(a_{j-1})}^{z(a_j)} \frac{\epsilon}{L} dz| = \epsilon.$$

In comparing $\int_a^b f(z(t))z'(t)dt$ with $\sum_{i=1}^{n} f(z(a_i))(z(a_i) - z(a_{i-1}))$, many students wrote something like $\lim_{|P| \to 0} \sum_{i=1}^{n} f(z(a_i))z'(a_i) = \int_C f dz$. There are a few things wrong with this. First, as $|P|$ (the mesh size of P) shrinks, n necessarily goes to infinity. A lot of students showed that the discrepancy in each subinterval between the Riemann summand and the integral goes to 0 as the mesh size shrinks, which is all nice and good, but if the number of subintervals goes to infinity while the error in each summand goes to 0, that doesn't help.

Second, the precise definition of a (real) Riemann integral is that the integral is equal to the infimum, over all partitions, of the upper Riemann sum (that is, the Riemann sum where one uses in each subinterval a sample point that maximizes f) and equal to the supremum, over all partitions, of the lower Riemann sum. (Of course, both of these quantities must be equal for the Riemann integral to be defined). The point here is that it is not good enough to consider only right-endpoint Riemann sums.

Consider the function $f : [0, 1] \to \mathbf{R}$, defined to be 1 when x is rational and 0 when x is irrational. It is possible to construct partitions of arbitrarily small mesh whose points are rational. The right-endpoint Riemann sum of f with respect to this partition will be 1. But if the partition has irrational points, then the right-endpoint Riemann sum of f will be the size of the last subinterval, which can be arbitrarily small. If f is continuous, then no such anomalies occur, and any sequence of progressively finer partitions will yield a sequence of Riemann sums converging to $\int f$, no matter where the sample points are taken in the partitions. This is proven using the uniform continuity argument above.

The cleanest way to solve this problem was to let the ML-inequality do the dirty work, as above.

Problem 2.

We know from lecture that if $f(z)$ is analytic on the closed disc $\{|z| \leq R\}$ then $\int_{|z|=R} f(z)dz = 0$. Show that this conclusion remains true if we just assume that $f(z)$ is continuous on the closed disc $\{|z| \leq R\}$ and analytic on the open disc $\{|z| \leq R\}$. [**Hint:** Approzimate $f(z)$ uniformly with the functions $f_r(z) = f(rz)$ for $0 < r < 1$.]

Let D be the disc $\{z \in \mathbb{C} | |z| < R\}$ and let $\bar{D}$ be the closure of D. f is analytic on $\bar{D}$ and continuous on D. Define functions $f_r : \bar{D} \to \mathbb{C}, f_r(z) = f(zr)$. The functions $f_r(z) = f(rz)$ converge pointwise to f uniformly as follows: Because f is continuous on a compact set, it is uniformly continuous which means that for each $\epsilon > 0$ there exists a δ such that for all $z_1, z_2 \in \bar{D}$ with $|z_1 - z_2| < \delta$, $|f(z_1) - f(z_2)| < \epsilon$. For r such that $|1 - \frac{r}{R}| < \delta$ we have that $|f(z) - f_r(z)| < \epsilon$, as desired i.e. the functions f_r converge to f uniformly.

$$\int_{|z|=R} f(z)dz = \int_{|z|=R} \lim_{r\to 1} f_r(z)dz$$

and by uniform convergence

$$\int_{|z|=R} \lim_{r\to 1} f_r(z)dz = \lim_{r\to 1} \int_{|z|=R} f_r(z)dz.$$

Now, each of the f_r is analytic on $\bar{D}$ because $f_r(z) = f(rz)$ so $f_r'(z) = f'(rz)$ which always exists when $r < 1$. Thus the closed curve theorem applies to each of the f_r and we have

$$\lim_{r\to 1} \int_{|z|=R} f_r(z)dz = \lim_{r\to 1} 0 = 0$$

and the theorem is proved.

Problem 3.

Evaluate the contour integral

$$\int_{|z|=3} \frac{dz}{1+z^2}$$

by using the Cauchy integral formula and partial fractions.

$$\int_{|z|=3} \frac{dz}{1+z^2} = \frac{i}{2} \int_{|z|=3} \frac{-1}{z-i} + \frac{1}{z+i} dz$$

by partial fractions. Then if we set $g(z) = 1$, we see from the Cauchy Integral Formula that $\int_{|z|=3} \frac{1}{z-a} dz = \int_{|z|=3} \frac{g(z)}{z-a} dz = 2\pi i g(a) = 2\pi i$. So we have

$$\frac{i}{2} \int_{|z|=3} \frac{-1}{z-i} + \frac{1}{z+i} dz = \frac{i}{2}(-2\pi i + 2\pi i) = 0.$$

Problem 4.

A function $f(z)$ on $\mathbb{C}$ is said to be *doubly periodic* if there exist complex numbers ω_1, ω_2 (called the *periods* of f), linearly independent over $\mathbb{R}$, such that

$$f(z+\omega_1) = f(z+\omega_2) = f(z)$$

for all $z \in \mathbb{C}$. Show that a doubly periodic entire function must be constant.

Suppose f is doubly periodic and entire with periods ω_1 and ω_2. The set $P = \{r_1\omega_1 + r_2\omega_2 \text{for} 0 \leq r_1, r_2 \leq 1\}$ (a closed parallelogram) is compact. Thus f has a maximum value M on P. ω_1 and ω_2 are linearly independent over $\mathbb{R}$, and $\mathbb{C}$ is of dimension 2 as a real vector space, so every element z of $\mathbb{C}$ is expressible as $a_1\omega_1 + a_2\omega_2$ for some $a_1, a_2 \in \mathbb{R}$. By the double periodicity, if $a_1 = r_1 \pmod 1$ and $a_2 = r_2 \pmod 1$ then $f(z) = f(a_1\omega_1 + a_2\omega_2) = f(r_1\omega_1 + r_2\omega_2)$. $r_1\omega_1 + r_2\omega_2$ is in P, so its value under f, and hence $f(z)$, is less than or equal to M. Thus f is bounded. Then by Liouville's theorem, f is constant.

Problem 5.

Recall the following statement of Green's theorem (which is slightly more general than the version presented in lecture). Let τ be a smooth, simple closed curve, oriented counterclockwise, which surrounds the region D in $\mathbb{R}^2$. Denote by $\bar{D}$ the closure of D, i.e., $\bar{D} = D \cup \tau$. let $P, Q : \bar{D} \to \mathbb{R}$ be continuously differentiable (C_1) real-valued functions. Then

$$\int_\tau P dx + Q dy = \int\int_D (Q_x - P_y) dx dy.$$

1. Use Green's theorem to prove that if gz is a function on $\bar{D} \subset \mathbb{C}$, then

$$\int_\tau g(z) dz = 2i \int\int_D \frac{\partial g}{\partial \bar{z}} dx dy.$$

   Explain why this formula implies the closed curve theorem if g is C^1 and entire.

2. Show that

$$\text{Area}(D) = \frac{1}{2i}\int_\tau \bar{z} dz = \int_\tau x dy.$$

3. Hiker Bob, walking on a sheet of graph paper, begins at the origin and then traces out a simple closed curve τ, always walking along the grid (i.e., always moving an integral number of units in one of the four coordinate directions). Bob moves a total of 6 units north along ponts with x-coordinate 3, 4 units north with x-coordinate 7, 2 units south with x-coordinate 2, and 8 units south with x-coordinate 0. Use part (b) to find the area of the region enclosed by τ.

1. Let gz be a function on $\bar{D} \subset \mathbb{C}$. Let $g(z) = u(z) + iv(z)$ where $u, v : \mathbb{C} \to \mathbb{R}$ which we also

think of as being from $\mathbb{R}^2$ to $\mathbb{R}$. Then

$$\begin{aligned}
\int_\tau g(z)dz &= \int_\tau (u(z)+iv(z))(dx+idy) \\
&= \left(\int_\tau u(z)dx - v(z)dy\right) + i\left(\int_\tau v(z)dx + u(z)dy\right) \\
&= \int\int_D (-v_x(z) - u_y(z))dxdy + i\int\int_D (u_x(z) - v_y(z))dxdy \\
&= \int\int_D (iu_x(z) - v_x(z) - u_y(z) - iv_y(z))dxdy \\
&= \int\int_D (i(u_x(z) + v_x(z)) - (u_y + iv_y(z)))dxdy \\
&= \int\int_D (ig_x(z) - g_y(z))dxdy \\
&= 2i\int\int_D \frac{1}{2}\left(g_x(z) - \frac{1}{i}g_y(z)\right) \\
&= 2i\int\int_D \frac{\partial g}{\partial \bar{z}}dxdy.
\end{aligned}$$

By homework 2, if g is entire then $\partial g/\partial\bar{z} = 0$ everywhere, so the right side is zero. Thus we get that when g is entire and τ is a smooth simple closed curve $\int_\tau g(z)dz = 0$, which is part of the closed curve theorem.

2. Set $g(z) = \bar{z}$. By part a, we have

$$\int_\tau \bar{z}dz = \int_\tau g(z)dz = \int\int_D \frac{\partial g(z)}{\partial\bar{z}} = \int\int_D \frac{\partial\bar{z}}{\partial\bar{z}} = \int\int_D dxdy = \text{Area}(D).$$

Or, set $P, Q : \mathbb{R}^2 \to \mathbb{R}$, $P(x,y) = 0, Q(x,y) = x$. Then $P_y = 0, Q_x = 1$.

$$\text{Area}(D) = \int\int_D 1dxdy = \int\int_D (Q_x - P_y)dxdy = \int_\tau Pdx + Qdy = \int_\tau 0 + xdy.$$

3. By b,

$$\text{Area}(D) = \int_\tau xdy$$

so this is just $6*3 + 4*7 + (-2)*2 + 8*0 = 42$.

Problem 6.

Let $f(z) = c_0 + c_1 z + \ldots + c_n z^n$ be a polynomial with $c_k \in \mathbb{C}$ for all k. Prove that

$$2|\int_{-1}^{1} f(x)^2| \le \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta = 2\pi\sum_{k=0}^{n} |c_k|^2.$$

[**Hint:** For the first inequality, apply the closed curve theorem to the function $f(z)^2$ on the top and bottom halves of the unit disc. For the second, first work out the value of $\int_0^{2\pi} e^{ik\theta}d\theta$ for all $k \in \mathbb{Z}$.]

Let curves C_1, C_2, C_3 be as in the diagram below. $f(z)^2$ is an entire function, so by the closed curve theorem

$$\int_{C_1+C_2} f(z)^2dz = \int_{C_1} f(z)^2dz + \int_{C_2} f(z)^2dz = 0$$

and

$$\int_{C_3-C_2} f(z)^2dz = \int_{C_1} f(z)^2dz - \int_{C_2} f(z)^2dz = 0.$$

Combining the two, we get

$$\begin{aligned}
2\int_2 f(z)^2dz &= \int_{C_3} f(z)^2dz - \int_{C_1} f(z)^2dz \\
2|\int_2 f(z)^2dz| &\leq |\int_{C_3} f(z)^2dz| + |\int_{C_1} f(z)^2dz| \\
&\leq \int_{C_3} |f(z)^2|dz + \int_{C_1} |f(z)^2|dz \\
&= \int_{C_1+C_3} |f(z)^2|dz \\
&= \int_0^{2\pi} |f(e^{i\theta})|^2d\theta
\end{aligned}$$

For the second half, we first prove that

$$\int_0^{2\pi} e^{ik\theta}d\theta = \begin{cases} 0, & \text{if } k \neq 0; \\ 2\pi, & \text{if } k = 0. \end{cases}$$

Proof. When $k = 0$, $\int_0^{2\pi} e^{ik\theta}d\theta = \int_0^{2\pi} d\theta = 2\pi$. When $k \neq 0$, $e^{ik\theta}$ is the derivative of $\frac{1}{ik}e^{ik\theta}$, so by the fundamental theorem of complex calculus,

$$\int_0^{2\pi} e^{ik\theta}d\theta = \frac{1}{ki}(e^{ik\theta}|_0^{2\pi} = \frac{1}{ki}(e^{ik2\pi} - e^0) = 0.$$

□

Now we note that

$$\begin{aligned}
\int_0^{2\pi} |f(e^{i\theta})|^2d\theta &= \int_0^{2\pi} f(e^{i\theta})\overline{f(e^{i\theta})}d\theta \\
&= \int_0^{2\pi} (c_0 + c_1e^{i\theta} + \ldots + c_ne^{in\theta})(\bar{c_0} + \bar{c_1}e^{-i\theta} + \ldots + \bar{c_n}e^{-in\theta})d\theta \\
&= \int_0^{2\pi} (|c_0| + |c_1| + \ldots + |c_n| + \text{a bunch more terms})d\theta
\end{aligned}$$

where the interesting thing to note about each of the remaining terms is that they are each a constant times $e^{ki\theta}$ for some non-zero k. Then by the fact proved above, the net contribution of all the other terms to the integral is zero, and we are left with

$$\int_0^{2\pi} (|c_0| + |c_1| + \ldots + |c_k| d\theta = 2\pi \sum_{k=0}^{n} |c_k|^2.$$

These solutions are adapted from Ken Ferry's with (a lot of) TeX support from Joe Rabinoff