Problem Set 4 Solution Set

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1. (a) Let n be a positive integer. Using Cauchy's integral formula, calculate the integral

$$\oint_C \left(z - \frac{1}{z}\right)^n \frac{dz}{z},$$

where C is the unit circle in $\mathbb{C}$.

Solution. Let $f(z) = (z^2 - 1)^n$. Since f is holomorphic and

$$\oint_C \left(z - \frac{1}{z}\right)^n \frac{dz}{z} = \oint_C \frac{(z^2-1)^n}{z^{n+1}}\, dz,$$

the Cauchy Integral Formula tells us that

$$\oint_C \left(z - \frac{1}{z}\right)^n \frac{dz}{z} = \frac{2\pi i f^{(n)}(0)}{n!}.$$

Now we expand f taking derivatives:

$$f(z) = (z^2-1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k z^{2n-2k},$$

so that

$$f^{(n)}(z) = \sum_{k=0}^{[n+1/2]} \binom{n}{k} (-1)^k (2n-2k)\cdots(n-2k) z^{n-2k}$$

This means

$$f^{(n)}(0) = \begin{cases} \binom{n}{n/2}(-1)^{n/2} n! & n \text{ is even,} \\ 0 & n \text{ is odd.} \end{cases}$$

Hence

$$\oint_C \left(z - \frac{1}{z}\right)^n \frac{dz}{z} = \begin{cases} 2\pi i \binom{n}{n/2}(-1)^{n/2} & n \text{ is even,} \\ 0 & n \text{ is odd.} \end{cases}$$

□

(b) By using the substitution $z \mapsto e^{it}$ in the integral above, evaluate

$$\int_0^{2\pi} \sin^n z\, dz.$$

Solution. The desired substitution in the above integral yields

$$\oint_C \left(z - \frac{1}{z}\right)^n \frac{dz}{z} = \int_0^{2\pi} (e^{it} - e^{-it})^n \frac{ie^it\,dt}{e^{it}}$$
$$= i \cdot (2i)^n \int_0^{2\pi} \sin^n t\,dt.$$

By part (a) we easily compute that

$$\int_0^{2\pi} \sin^n t\,dt = \begin{cases} \dfrac{\pi}{2^{n-1}}\dbinom{n}{n/2} & n \text{ is even,} \\ 0 & n \text{ is odd.} \end{cases}$$

□

2. Let τ be a complex number that is not real. Let $f(z)$ be a holomorphic function such that $f(z+1) = f(z)$ and $f(z+\tau) = f(z)$. Prove that f is constant.

   Solution. Ah...elliptic functions. The two conditions on f tell us that this function is "doubly periodic". Put another way, the values of f in $\mathbb{C}$ are determined by the values f takes on the parallelogram spanned by the origin, 1, τ and $1+\tau$. (Note this is a non-degenrate parallelogram since τ is not real.) But the parallelogram is a compact region in $\mathbb{C}$. Hence f achieves a maximum M in this region and by translation $f(z) \leq M$ for all $z \in \mathbb{C}$. Since f is entire (i.e., holomorphic in all of $\mathbb{C}$) and bounded, Liouville's theorem tells us f is constant. □

3. Let $P(z) = a_0 + a_1 z + \cdots + a_n z^n$, where $a_n \neq 0$. Show there exist n complex numbers $\alpha_1, \ldots, \alpha_n$, possibly not distinct, such that

$$P(z) = a_n(z - \alpha_1)\cdots(z - \alpha_n).$$

   Solution. We use induction on n. The case $n = 0$ is clear. Suppose the claim holds for all polynomials of degree less than or equal to $n-1$. Let $P(z)$ be a polynomial of degree n as above. Then by the fundamental theorem of algebra $P(z)$ has a root. Call this root α_n. Then the division algorithm (which holds in $\mathbb{C}$) tells us that

$$P(z) = (z - \alpha_n)Q(z) + R(z),$$

   where R in this case is constant. Since $P(\alpha_n) = 0$, it follows that $R(z) = R(\alpha_n) = 0$. Note that the leading coefficient of Q is still a_n and that Q has degree $n-1$. By inductive hypothesis there exist $n-1$ possibly non-distinct numbers $\alpha_1, \ldots, \alpha_{n-1}$ such that

$$Q(z) = a_n(z - \alpha_1)\cdots(z - \alpha_{n-1}).$$

   Whence

$$P(z) = a_n(z - \alpha_1)\cdots(z - \alpha_n).$$

□

4. Let C be the circle $|z| = 2$ in $\mathbb{C}$. Evaluate the integral

$$\frac{1}{2\pi i} \oint_C f(z)\, dz$$

for the following functions $f(z)$. Here $k \in \mathbb{N}$.

Solution. I'll do a couple of the problematic integrals. I myself messed up in part (p). If you want your point back send me an email and I'll gladly give it to you. □

(m) $(z - \sin z)/(z^2 \sin z)$

Solution. Zero is the only singularity of the function inside C. Now note that $\sin z$ has a simple zero at 0. Hence the residue at 0 of this function is just

$$\frac{1}{2} \lim_{z \to 0} \frac{d^2}{dz^2} \left(\frac{z(z - \sin z)}{\sin z} \right) = 0.$$

Hence

$$\frac{1}{2\pi i} \oint_C f(z)\, dz = \sum \text{Res} \; = 0.$$

□

(p) $e^{1/z}/(1 - z)$

Solution. A problem like this should begin to ring "inversion" in your mind. (It didn't in mine at first...) The trick is to use the substitution $z \mapsto -1/t$. This takes the unit disk into the outside world and viceversa. The minus sign is there to preserve the orientation (counterclockwise) of the image of the contour C. Under this map C maps to $|z| = 1/2 =: C'$. Hence

$$\frac{1}{2\pi i} \oint_C \frac{e^{1/z}}{1 - z}\, dz = \frac{1}{2\pi i} \oint_{C'} \frac{e^{-t}}{1 - 1/t} \cdot \frac{-1}{t^2}\, dt = \frac{1}{2\pi i} \oint_{C'} \frac{e^{-t}}{t(1 - t)}\, dt.$$

The only singularity inside C' is $t = 0$. The residue at 0 is just

$$\lim_{t \to 0} \frac{t e^{-t}}{t(1 - t)} = -1.$$

Hence

$$\frac{1}{2\pi i} \oint_C \frac{e^{1/z}}{1 - z}\, dz = \frac{1}{2\pi i} \oint_{C'} \frac{e^{-t}}{t(1 - t)}\, dt = \sum \text{Res} \; = -1.$$

□

(u) $\tan z/z^2$

Solution. Note that $f(z) = \sin z/(z^2 \cos z)$. The singularities of this function inside C lie at $-\pi/2, 0$ and $\pi/2$. The cosine function has simple zeroes at $-\pi/2$ and $\pi/2$. The function has a pole of order 2 at 0, hence the residue at the origin is

$$\lim_{z\to 0} \frac{d}{dz}\left(\frac{z^2 \sin z}{z^2 \cos z}\right) = 1.$$

The residues at $-\pi/2$ and $\pi/2$ are, respectively,

$$\lim_{z\to\pi/2} \frac{(z-\pi/2)\sin z}{z^2 \cos z} \quad \text{and} \quad \lim_{z\to-\pi/2} \frac{(z+\pi/2)\sin z}{z^2 \cos z}.$$

Using L'Hôpital's rule we see that both these expressions evaluate to $-4/\pi^2$ Hence

$$\frac{1}{2\pi i}\oint_C \frac{\tan z}{z^2}\, dz = \sum \text{Res} \; = 1 - \frac{4}{\pi^2} - \frac{4}{\pi^2} = 1 - \frac{8}{\pi^2}.$$

□