Deforming Complex Integrals

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We have already seen that under many circumstances the complex integral of an analytic function along a path does not depend on the specific path taken, but only on the endpoints of the path, and correspondingly the integral around a closed curve is 0. Intuitively, the complex integrals along two different paths are equal if we can deform one path to the other smoothly, without getting caught up on singularities where the function is not defined. Let us formalise these intuitions.

1 Deforming a curve

Intuitively, a "deformation" of a curve in a region R should be a continuous path in the space of paths: if I is the unit interval $[0, 1]$ and $I \rightsquigarrow \mathbb{C}$ is the space of piecewise-smooth paths from $z \in R$ to $w \in R$, then we are interested in continuous maps

$$I \to (I \rightsquigarrow R)$$

from the interval, to the space of paths. To make this precise, we would need to put a topology on the space of piecewise-smooth paths, which would take us too far afield. Instead, we will use a general fact: the set of functions from A to (functions from B to C),

$$A \to (B \to C),$$

is isomorphic to the set of functions from $A \times B$ to C,

$$A \times B \to C.$$

We can therefore define deformations in the space of paths in terms of maps from the square $I \times I$ to R. The formal name for this type of deformation is a *homotopy*.

Definition 1. *A homotopy between two piecewise smooth paths Γ_0 and Γ_1, each starting at $z \in R$ and ending at $w \in R$, is a map*

$$\Phi\colon I \times I \to R$$

so that:

- *Φ is continuous;*
- *The map $s \mapsto \Phi(0, s)$ agrees with the path Γ_0;*
- *The map $s \mapsto \Phi(1, s)$ agrees with the path Γ_1; and*
- *For each fixed $t \in I$, the map $s \mapsto \Phi(t, s)$ is a piecewise smooth path Γ_t from z to w.*

If there is a homotopy between Γ_0 and Γ_1, they are said to be homotopic within R.

That is, on each horizontal line through the square, we see a path from z to w; as we move the line from top to bottom in the square, the path is deformed from Γ_0 to Γ_1.

(It is a little asymmetric that we require the paths, the horizontal lines in the square, to be piecewise smooth, but only require the map to be continuous in the vertical direction. We could require the map Φ itself to be piecewise smooth in some sense, but as we will see, that will not be necessary.)

The key fact for our purposes is that if Γ_0 and Γ_1 are homotopic, then the complex integral along Γ_0 equals that along Γ_1.

Proposition 2. *If $f(z)$ is an analytic function defined on a region R, and Γ_0 and Γ_1 are homotopic within R, then*

$$\int_{\Gamma_0} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz.$$

The proof of Proposition 2 will rely on covering the square defining the homotopy by little rectangles, so that the image of each rectangle is contained inside a rectangle; since we have already proved that complex integrals do not depend on the path within a rectangle, we will be able to succesively push the paths across the rectangles. But first, let's look at a somewhat easier problem, one dimension down.

Proposition 3. *If Γ is a piecewise-smooth path from z to w within a region R, then there is a piecewise-linear path Γ' (with the same endpoints as Γ) so that, for all analytic functions $f(z)$ defined on R,*

$$\int_{\Gamma} f(z)\,dz = \int_{\Gamma'} f(z)\,dz.$$

(A *piecewise-linear* path is a concatenation of straight line segments.)

Proof. Since R is a region, each point $z(t)$ on Γ is contained in a disk of some positive radius $r(t)$ which is contained inside R. Since the map $t \mapsto z(t)$ defining Γ is a continuous map, for each t there is some interval $I(t)$ centered on t so that $I(t)$ is mapped inside the disk of radius $r(t)$. These intervals give a covering of the interval defining Γ, which is compact, so there is a finite sub-covering. Order the intervals

$$0 \in I_0, I_1, \ldots, I_n \ni 1$$

so that $I_{k-1} \cap I_k \neq \emptyset$.

Pick a point t_k inside each $I_{k-1} \cap I_k$; set $t_0 = 0$ and $t_{n+1} = 1$. Set $z_k = z(t_k)$ (so that $z_0 = z$ and $z_{n+1} = w$). Then t_k and t_{k+1} are both contained inside I_k, and so by the choice of I_k, we see that z_k, z_{k+1}, and the portion of Γ between them are all contained inside a single disk. Since the disk is convex, the straight line segment between them is also contained inside the disk. But we have already seen that inside a disk (on which $f(z)$ is defined) the complex integral between the points z_k and z_{k+1} does not depend on the path we take, so we can replace the portion of the piecewise-smooth path Γ between z_k and z_{k+1} by this straight line segment. Repeating this for each sub-interval, we replace Γ by a piecewise linear path Γ' which runs through the points

$$z_0 — z_1 — \cdots — z_{n-1} — z_n$$

in sequence. □

As an aside, we can use the ideas in Proposition 3 to *define* the complex integral along an arbitrary continuous curve, not necessarily piecewise-smooth, even though the naïve Riemann integral does not necessarily converge along such a curve.

We are now ready to prove Proposition 2.

Proof. For a fixed t, consider the curve Γ_t. As in Proposition 3, we can cover the interval I with an overlapping sequence of intervals I_k so that each I_k is mapped by $\Phi(t, \cdot\,)$ into a disk $D_k \subset R$. Let $z_k \in D_k \cap D_{k+1}$ be a point on Γ_t.

Note that, since the map Φ is a continuous function, there will be some open interval $(t-\varepsilon, t+\varepsilon)$ so that for each $t' \in (t-\varepsilon, t+\varepsilon)$, $\Gamma_{t'}$ satisfies the same conditions as Γ_t: each I_k is mapped by $\Phi(t', \cdot\,)$ into the same disk D_k. Let $z_k' \in D_k \cap D_{k+1}$ be the corresponding points on $\Gamma_{t'}$.

Now we can use the invariance of complex integrals within a disk to show that the integral along Γ_t equals the integral along $\Gamma_{t'}$: the integral along Γ_t is, as before, equal to the integral along the piecewise linear path

$$z = z_0 \text{---} z_1 \text{---} z_2 \text{---} \cdots \text{---} z_{n-1} \text{---} z_n = w.$$

Since $z_0 = z_0'$, z_1, and z_1' are all in the same disk D_1, this is also equal to the integral along the piecewise linear path

$$z = z_0' \text{---} z_1' \text{---} z_1 \text{---} z_2 \text{---} \cdots \text{---} z_{n-1} \text{---} z_n = w.$$

Since z_1, z_1', z_2, and z_2' are all in the same disk D_2, this is equal to the integral along the path

$$z = z_0' \text{---} z_1' \text{---} z_2' \text{---} z_2 \text{---} \cdots \text{---} z_{n-1} \text{---} z_n = w.$$

Continuing in this fashion, we eventually show that the integral along Γ_t is the same as the integral along

$$z = z_0' \text{---} z_1' \text{---} z_2' \text{---} \cdots \text{---} z_{n-1}' \text{---} z_n' = w$$

which is also equal to the integral along $\Gamma_{t'}$.

Therefore the integral along Γ_t is equal to the integral along $\Gamma_{t'}$ for all t' in a neighborhood of t. But since this was true for all t and the interval is connected, the integral along all of the different Γ_t must be equal; in particular,

$$\int_{\Gamma_0} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz.$$ □

2 Simple connectivity

Of particular interest are those regions R so that the complex integral between two points is completely independent of the path. Based on the above discussion, there is a clear condition for this to be true.

Definition 4. *A region R is said to be simply connected if, for every pair of points z, w in R and every pair of piecewise-smooth paths Γ_0, Γ_1 between z and w, the paths Γ_0 and Γ_1 are homotopic.*

The intuition behind the name is that an open subset of $\mathbb{C}$ is *connected* if there is a path between any pair of points z, w; it is *simply connected* if there is essentially only one way to connect z and w (up to homotopy).

A complete discussion of simple connectivity is outside the scope of this course. Let us content ourselves with a few observations. The first is the reason we defined simple connectivity:

Proposition 5. *In a simply connected region R, the integral $\int_\Gamma f(z)dz$ of an analytic function along a path Γ depends only on the endpoints of Γ. In particular, if Γ is a closed curve, the integral along Γ is 0.*

The proof is immediate from the definitions.

With a little bit of thought, you should be able to convince yourself that the number of ways to connect a pair of points z and w is independent of z and w; that is, in Definition 4 we can pick the pair of points z and w rather than quantifying over all pairs. A standard choice is to pick $z = w$. In this case, there is a canonical path from z to z: the constant path. (Strictly speaking, this is not a piecewise-smooth path according to our earlier definition, since the derivative is not 0. We need to modify the notion of homotopy slightly: we need to drop the requirement that $\Phi(t, \cdot\,)$ is piecewise-smooth when $t = 1$. In fact, we could equally well drop this requirement for all t and get an equivalent definition of simple-connectivity, since we have already seen how to "approximate" an arbitrary continuous curve by a piecewise-linear curve.)

Definition 6. *A region R is said to be simply-connected if, for some point z in R, every path from z to itself is homotopic to the constant path. In other words, R is simply-connected if, for every Γ from z to z, there is a map*

$$\Phi: I \times I \to R$$

so that

- *Φ is continuous;*
- *$\Phi(0, \cdot\,)$ coincides with the path Γ;*
- *$\Phi(t, 0) = \Phi(t, 1) = z$ (in other words, $\Phi(t, \cdot\,)$ is a path from z to itself);*
- *$\Phi(1, s) = z$; and*
- *(optional) $\Phi(t, \cdot\,)$ is piecewise-smooth for each t.*

The definition is true for the same regions R whether or not we include the last condition.

(The difference between Definition 6 and Definition 4 is precisely the difference between showing that the integral along a path is independent of the path, and showing that the integral along any closed curve is 0.)

The advantage of this definition over the earlier one is that it is easier to show that certain regions are simply-connected.

Proposition 7. *Any convex region R is simply connected.*

Proof. Take any path $z(s)$ from z_0 to itself, and define the homotopy Φ by

$$\Phi(t, s) = (1 - t)\, z(s) + t\, z_0$$

That is, the path $\Phi(t, \cdot\,)$ interpolates linearly between the path $z(s)$ at $t = 0$ and the constant path at z_0 at $t = 1$. Since R is convex, the line between each point $z(s)$ and z_0 is contained inside of R, so Φ is a continuous map from $I \times I$ to R. The remaining properties of Φ are immediate. □

This proof applies to a slightly more general class of regions: those which are *star-shaped*. A region is said to be star-shaped if there is a point $z_0 \in R$ so that, for every $w \in R$, the straight line segment between z_0 and w is contained inside R. But many more regions are simply-connected than just the star-shaped ones.

In the Bak and Newman text there is a completely different definition of simply connected for regions in $\mathbb{C}$. There definition turns out to be equivalent to the one above, although that is not obvious and the definition above generalises to more general spaces, while the one in Bak and Newman does not.

Finally, let me mention the following result, the converse to Proposition 5:

Theorem 8. *Let R be a region in $\mathbb{C}$. If, for every analytic function $f(z)$ defined on R and every closed curve Γ in R, $\int_\Gamma f(z) dz = 0$, then R is simply connected.*