Math 113 Problem Set 9

Due Monday, November 17, 2003

1. Needham, Chapter 3, Page 184, problem 14.
2. Needham, Chapter 3, Page 185, problem 15.
3. For each of the following regions R, find an explicit map from R to the unit disk.
   (a) The complement of a semicircle: $\{z \in \mathbb{C} : |z| > 1\} \cup \{z \in \mathbb{C} : \operatorname{Im} z < 0\} \cup \{\infty\}$.
   (b) The upper half plane with a slit removed: $\{z \in \mathbb{C} : \operatorname{Im} z > 0\} \setminus [0, i]$.
   (c) Half of an infinite strip: $\{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1, \operatorname{Im} z > 0\}$.
   (d) (Optional) The upper half plane with slits $[n, n+i]$ extending upwards from each integer removed: $\{z \in \mathbb{C} : \operatorname{Im} z > 0\} \setminus \bigcup_{n\in\mathbb{Z}} [n, n+i]$. (Hint: using the previous part, map the strip to itself so the corners go to points along the vertical side, and apply the Reflection Principle.)
4. (a) Show that under a change of coordinates

$$z \mapsto w = g(z)$$

   the residue of a meromorphic differential $f(z)\,dz$ does not change. In fact, show that the residue is the only coefficient in the Laurent expansion for $f(z)$ that does not change when we change coordinates. (That is, if you expand $f(z)$ as a Laurent series in z, or if you transform the transformed $\tilde{f}(w)\,dw$ as a Laurent series in w, only the coefficients that correspond to the residue match.)

   As a result, it's proper to talk about the residue of a differential, and not the residue of a function, but usually this distinction is ignored.

   (b) Show that, for any meromorphic function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, the sum of the residues at all poles of $f(z)\,dz$ is equal to 0. (Hint: you might want to change coordinates to $w = 1/(z - z_0)$, where z_0 is neither a pole nor a zero of f, and integrate around a large loop.)

   (c) For any meromorphic function f, compute the residues of the differential

$$\frac{f'(z)\,dz}{f(z)}$$

   to give another proof that the number of zeros of f equals the number of poles.