Solutions for Problem Set #1
due September 19, 2003
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(B&N 1.4) *Prove the following identities: (2.5 points each)*

a. $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$

Write $z_1 = a_1 + b_1 i, z_2 = a_2 + b_2 i$, where a_1, b_1, a_2, b_2 are real numbers. Thus,

$$\begin{aligned} \overline{z_1 + z_2} &= \overline{(a_1 + a_2) + (b_1 + b_2)i} \\ &= (a_1 + a_2) + (-b_1 - b_2)i \\ &= a_1 - b_1 i + a_2 - b_2 i \\ &= \overline{z}_1 + \overline{z}_2 \end{aligned}$$

b. $\overline{z_1 z_2} = \overline{z}_1 \cdot \overline{z}_2$

With a_1, b_1, a_2, b_2 defined as above:

$$\begin{aligned} \overline{z_1 z_2} &= \overline{(a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i} \\ &= (a_1 a_2 - b_1 b_2) + (-a_1 b_2 - a_2 b_1)i \\ &= (a_1 - b_1 i) \cdot (a_2 - b_2 i) \\ &= \overline{z}_1 \cdot \overline{z}_2 \end{aligned}$$

c. $\overline{P(z)} = P(\overline{z})$

Let $P(z) = a_n z^n + \ldots + a_0$, where a_j is real for all $0 \leq j \leq n$. First note that since raising to an integer power is just repeated multiplication, repeated application of our result from part b, proves that $\overline{z^n} = \overline{z}^n$

$$\begin{aligned} \overline{P(z)} &= \overline{a_n z^n + \ldots + a_0} \\ &= \overline{a_n z^n} + \ldots + \overline{a}_0 \text{ (by part a)} \\ &= \overline{a}_n \overline{z^n} + \ldots + \overline{a}_0 \text{ (by part b)} \\ &= a_n \overline{z^n} + \ldots a_0 \text{ each } a_k \text{ is real} \\ &= a_n \overline{z}^n + \ldots a_0 \text{ (by above)} \\ &= P(\overline{z}) \end{aligned}$$

d. $\overline{\overline{z}} = z$

Write $z = a + bi$, as usual, and then

$$\overline{\overline{z}} = \overline{a - bi} = a + bi = z$$

(B&N 1.5) *Suppose P is a polynomial with real coefficients. Show that $P(z) = 0$ if and only if $P(\overline{z}) = 0$. (5 points)*

$P(z) = 0$ is true iff $\overline{P(z)} = \overline{0} = 0$. By 4c, the left hand side is equal to $P(\overline{z})$, and so $P(z) = 0$ iff $P(\overline{z}) = 0$.

(B&N 1.10) *Solve the following equations in polar form and locate the roots in the complex plane: (5 points each)*

a. $z^6 = 1$

Let $z = r \operatorname{cis} \theta = re^{i\theta}$. We know that

$$z^6 = r^6 \operatorname{cis} 6\theta = 1 = \operatorname{cis} 0$$

Therefore, $r^6 = 1$ implies that $r = 1$, to keep r positive. $\operatorname{cis} 6\theta = \operatorname{cis} 0$ means that $6\theta = 2\pi k$ for any integer k, so $\theta = \pi k/3$. Ignoring duplicates, the solutions are:

- $\operatorname{cis} 0 = 1$
- $\operatorname{cis} \frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2} i$
- $\operatorname{cis} \frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$
- $\operatorname{cis} \pi = -1$
- $\operatorname{cis} \frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$
- $\operatorname{cis} \frac{5\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2} i$

Geometrically, these solutions form a regular hexagon centered around the origin.

b. $z^4 = -1$

In polar form, the equation becomes:

$$z^4 = r^4 \operatorname{cis} 4\theta = -1 = \operatorname{cis} \pi$$

Again, we choose the positive root of $r = 1$, and $4\theta = \pi + 2\pi k$ for any integer k. Therefore, $\theta = \pi/4 + \pi k/2$. The four solutions can thus be written:

- $\operatorname{cis} \frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$
- $\operatorname{cis} \frac{3\pi}{4} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$
- $\operatorname{cis} \frac{5\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$
- $\operatorname{cis} \frac{7\pi}{4} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$

Geometrically, these form a square, centered at the origin, and whose sides are parallel to the real and imaginary axes.

c. $z^4 = -1 + \sqrt{3} i$

In polar form, the right hand side has modulus $\sqrt{(-1)^2 + \sqrt{3}^2} = \sqrt{4} = 2$, and argument $2\pi/3$ because this satisfies both $\cos\theta = -1/2$ and $\sin\theta = \sqrt{3}/2$. Thus,

$$z^4 = r^4 \operatorname{cis} 4\theta = 2 \operatorname{cis} \frac{2\pi}{3}$$

We solve for $r = \sqrt[4]{2}$, and from $\operatorname{cis} 4\theta = \operatorname{cis} \frac{2\pi}{3}$, we know that

$$4\theta = \frac{2\pi}{3} + 2\pi k$$

and thus

$$\theta = \frac{\pi}{6} + \frac{\pi k}{2}$$

Explicitly, the four solutions are:

- $\sqrt[4]{2} \operatorname{cis} \frac{\pi}{6} = \sqrt[4]{2} \left(\frac{\sqrt{3}}{2} + \frac{1}{2} i \right)$
- $\sqrt[4]{2} \operatorname{cis} \frac{2\pi}{3} = \sqrt[4]{2} \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i \right)$
- $\sqrt[4]{2} \operatorname{cis} \frac{7\pi}{6} = \sqrt[4]{2} \left(-\frac{\sqrt{3}}{2} - \frac{1}{2} i \right)$
- $\sqrt[4]{2} \operatorname{cis} \frac{5\pi}{3} = \sqrt[4]{2} \left(\frac{1}{2} - \frac{\sqrt{3}}{2} i \right)$

These form a square centered at the origin, but tilted.

(Needham 1.1) *The roots of a general cubic equation in X may be viewed (in the XY-plane) as the intersections of the X-axis with the graph of the cubic of the form*

$$Y = X^3 + AX^2 + BX + C$$

(i) Show that the point of inflection of the graph occurs at $X = -\frac{A}{3}$. (2.5 points)

The inflection point is the value of X where the second derivative changes sign. The second derivative is

$$\frac{d}{dX}3X^2 + 2AX + B = 6X + 2A$$

Solving this for 0 gives $X = -\frac{A}{3}$, and since the second derivative has constant positive slope, it is indeed positive for greater values of X and negative for smaller values.

(ii) *Deduce (geometrically) that the substitution $X = (x - \frac{A}{3})$ will reduce the above equation to the form $Y = x^3 + bx + c$. (2.5 points)*

Because the inflection point is at $X = -\frac{A}{3}$, this substitution will move the inflection point to $x = 0$, but from part (i), we know that the inflection point is proportional to the coefficient of the second-degree term, so if the former is zero, the latter must be as well. Therefore the equation has no x^2 term, and is therefore of the form mentioned.

(iii) *Verify this by calculation. (5 points)*

$$\begin{aligned}
Y &= X^3 + AX^2 + BX + C \\
&= (x - \frac{A}{3})^3 + A(x - \frac{A}{3})^2 + B(x - \frac{A}{3}) + C \\
&= x^3 - Ax^2 + \frac{A^2x}{3} - \frac{A^3}{27} \\
&\quad +Ax^2 - \frac{2A^2x}{3} + \frac{A^3}{9} \\
&\quad +Bx - \frac{BA}{3} + C \\
&= x^3 + (-A + A)x^2 + (\ldots)x + (\ldots) \\
&= x^3 + bx + c
\end{aligned}$$

for appropriate coefficients b and c.

(Needham 1.2) *Solve the cubic equation $x^3 = 3px + 2q$ using the inspired substition $x = s + t$. (15 points)*

(Note: I graded this problem as a single unit, and I am writing it up as a single unit as well. In the future, feel free to write up multi-part problems like this as a single thread of argumentation if it is more natural that way.)

As suggested, write $x = s + t$, and thus the equation becomes:

$$\begin{aligned}
(s+t)^3 &= 3p(s+t) + 2q \\
s^3 + 3s^2t + 3st^2 + t^3 &= 3p(s+t) + 2q \\
3st(s+t) + s^3 + t^3 &= 3p(s+t) + 2q
\end{aligned}$$

When written this way, this suggests breaking the left and right sides apart to yield the following pair of equalities:

$$\begin{aligned}
st &= p \\
s^3 + t^3 &= 2q
\end{aligned}$$

which are sufficient, but NOT necessary conditions for finding a solution to the previous equation.

Now we try to combine these two equations and t. Multiply the second equation by s^3 to get

$$s^6 + s^3t^3 = 2qs^3$$

and substituting the cube of the first equation eliminates t:

$$s^6 + p^3 = 2qs^3$$

We can rewrite this as a quadratic in s^3:

$$(s^3)^2 - 2q(s^3) + p^3 = 0$$

and apply the quadratic formula to get:

$$\begin{aligned} s^3 &= \frac{2q \pm \sqrt{4q^2 - 4p^3}}{2} \\ &= q \pm \sqrt{q^2 - p^3} \end{aligned}$$

Now we apply the equation $s^3 + t^3 = 2q$ to get that $t^3 = q \mp \sqrt{q^2 - p^3}$. Finally

$$\begin{aligned} x &= s + t \\ &= \sqrt[3]{q \pm \sqrt{q^2 - p^3}} + \sqrt[3]{q \mp \sqrt{q^2 - p^3}} \\ &= \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}} \end{aligned}$$

by the symmetry of the equation.

(Needham 1.4) *Show that if two integers can be expressed as the sum of two squares, then so can their product*

Suppose $M = a^2 + b^2$ and $N = c^2 + d^2$, where all the quantities are integers. Consider the complex numbers $z_1 = a + bi$ and $z_2 = c + di$. Notice that $|z_1|^2 = z_1\overline{z}_1 = M$ and $|z_2|^2 = z_2\overline{z}_2 N$. Therefore,

$$\begin{aligned} |z_1 z_2|^2 &= z_1 z_2 \overline{z_1 z_2} \\ &= z_1 \overline{z}_1 z_2 \overline{z}_2 \\ &= |z_1|^2 |z_2|^2 \\ &= MN \end{aligned}$$

But, we could also write $|z_1 z_2|^2$ as:

$$\mathrm{Re}(z_1 z_2)^2 + \mathrm{Im}(z_1 z_1)^2$$

Since $\mathrm{Re}(z_1 z_2)$ and $\mathrm{Im}(z_1 z_1)$ are integers, then MN is the sum of squares.

(B&N 1.11) *Show that the n-th roots of 1 (aside from 1) satisfy the cyclotomic equation:*

$$z^{n-1} + z^{n-2} + \ldots + z + 1 = 0$$

(5 points extra credit)

From the Fundamental Theorem of Algebra, we know we can write:

$$z^n - 1 = (z - a_1)(z - a_2) \ldots (z - a_n)$$

where the a_j are roots of $z^n - 1$, counting multiplicity. Of course, we know that 1 is a root, so WLOG, $a_n = 1$. Now we can also factor $z^n - 1$ in the following way:

$$\begin{aligned} (z-1)(z^{n-1} + \ldots + 1) &= z^n + z^{n-1} + \ldots + z \\ &\quad - z^{n-1} - \ldots - 1 \\ &= z^n - 1 \end{aligned}$$

Therefore, equating these two quantities and canceling the factor of $(z - 1) = (z - a_n)$, we get

$$z^{n-1} + \ldots + 1 = (z - a_1)(z - a_2) \ldots (z - a_{n-1})$$

and in particular, the non-unity roots nth roots of 1 are also roots of $z^{n-1} + z^{n-2} + \ldots + 1$.

(B&N 1.12) *Consider the $n - 1$ diagonols of a regular n-gon inscribed in a unit circle obtained by connecting one vertex with all the others. Show that the product of their lengths is n. (5 points extra credit)*

Consider the n-gon inscribed in the unit circle in the complex plane, with the one vertex as 1, and the other vertices are all the other nth roots of unity. Let $a_1 \ldots a_{n-1}$ denote the nth roots of unity other than n. Then, we can write the desired value as the product

$$|1 - a_1||1 - a_2| \ldots |1 - a_{n-1}|$$

If we substitute z for 1, then this equals:

$$\begin{aligned} &= |z - a_1| \ldots |z - a_{n-1}| \\ &= |(z - a_1) \ldots (z - a_{n-1})| \\ &= |z^{n-1} + \ldots + 1| \end{aligned}$$

by the previous problem. Of course, by substituting $z = 1$, this evaluates to n.