Math 113 Final Exam

Due January 16, 2004, at 5:00 PM

This exam is open notes and open book: you may use any of your notes and the textbooks for this course. You may also use a reference on real analysis if you wish. (Please indicate on the exam if you do so.) No other aids are permitted, and you may not discuss the content of the exam with anyone.

Each problem is worth 20 points. Your best 5 out of the 6 problems will count.

There will be a set of hints posted at 10PM today (Wednesday). Which problems get hints will be determined in part by you, the students; please e-mail me to nominate which problems you don't know how to get started on, or where else you got stuck.

1. Consider the function

$$f(z) = \frac{1}{1 - z - z^2} = \sum_{n=0}^{\infty} c_n z^n.$$

(a) Show that $c_{n+2} = c_{n+1} + c_n$ for $n \geq 0$. (The sequence c_n is called the *Fibonacci sequence.*)

Answer. Multiplying $f(z)$ by $1 - z - z^2$, we have

$$\begin{aligned} 1 + (z + z^2) f(z) &= f(z) \\ \sum_{n=0}^{\infty} c_n (z^{n+1} + z^{n+2}) &= \sum_{n=0}^{\infty} c_n z^n \\ 1 + c_0 z + \sum_{n=2}^{\infty} (c_{n-2} + c_{n-1}) z^n &= \sum_{n=0}^{\infty} c_n z^n \end{aligned}$$

Equating coefficients for c_n for $n \geq 2$, we get the desired equality. □

(b) What is the radius of convergence of the series?

Answer. $f(z)$ has a pole when its denominator has a root; the roots of the denominator are at

$$\frac{-1 \pm \sqrt{5}}{2}.$$

The closest root to 0 is at $(-1 + \sqrt{(5)})/2$, which is also the radius of convergence of the series. □

(c) Deduce a consequence of your answer in (b) for the Fibonacci sequence.

Answer. We know that the radius of convergence R can be computed by the limit

$$\limsup_{n\to\infty} |c_n|^{1/n} = 1/R = \frac{1+\sqrt{5}}{2}.$$

Since the c_n are positive and increasing, this is also the asymptotic growth rate

$$\lim_{n\to\infty} c_n^{1/n} = \frac{1+\sqrt{5}}{2}.$$

□

(d) (10 points) Consider the series a_n defined by

$$\begin{aligned} a_0 &= 1 \\ a_n &= \frac{a_{n-1}}{1} + \frac{a_{n-2}}{2} + \cdots + \frac{a_0}{n}, \qquad n > 0 \end{aligned}$$

Find $\lim_{n\to\infty} \frac{a_n}{a_{n-1}}$.

Answer. Define

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

First let's check that $f(z)$ has a non-zero radius of convergence, by showing that $a_n \leq 2^n$. We'll prove this by induction:

$$\begin{aligned} a_0 &= 1 = 2^0 \\ a_n &= \sum_{k=1}^{n} \frac{a_{n-k}}{k} < \sum_{k=1}^{n} a_{n-k} \leq \sum_{k=1}^{n} 2^{n-k} = 2^n - 1 < 2^n. \end{aligned}$$

Now recall that the power series expansion for $\log(1-z)$ is

$$\log(1-z) = \sum_{n=1}^{\infty} -\frac{z^n}{n}$$

Consider the product $-f(z)\log(1-z)$. The coefficient of z^n in the expansion of this product is the right hand side of the defining identity for a_n. We therefore have

$$\begin{aligned} -f(z)\log(1-z) &= \sum_{n=1}^{\infty} a_n z^n = f(z) - 1 \\ f(z) &= \frac{1}{1+\log(1-z)}. \end{aligned}$$

$f(z)$ has a singularity when $\log(1-z) = -1$ or $z = (e-1)/e$; this is the nearest singularity to 0, so the radius of convergence is $R = (e-1)/e$. By the ratio test, if $\lim_{n\to\infty} \frac{a_n}{a_{n-1}}$ exists, it is equal to $1/R$. To check that it exists, notice that the singularity of $f(z)$ at $z = (e-1)/e$ is a simple pole with residue $1/e$, so that

$$g(z) = f(z) - \frac{1}{ez - e + 1} = \sum_{n=0}^{\infty} b_n z^n$$

has no singularity at $z = (e-1)/e$ and has a strictly larger radius of convergence of 1. Thus

$$a_n = \frac{e^{n-1}}{(e-1)^n} + b_n$$

where

$$\limsup_{n\to\infty} |b_n|^{1/n} = 1.$$

In particular, this implies that

$$\lim_{n\to\infty} \frac{a_n}{a_n - 1} = \frac{e}{e-1}.$$

□

2. Let D be the unit disk $|z| < 1$ and let $f : D \to \mathbb{C}$ be an analytic function on D with a continuous extension to the boundary, and suppose that f is not zero on the boundary. Show that

$$\max_{z\in D} \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \# \text{ of zeros of } f \text{ in } D.$$

Answer. The number of zeros of f in D is given by

$$N = \frac{1}{2\pi i} \oint \frac{f'(z)}{f(z)}\, dz$$

where the integral runs around the boundary of D. In terms of a parametrization of the boundary by $z = e^{i\theta}$, this can be written

$$N = \frac{1}{2\pi} \int_0^{2\pi} \frac{f'(e^{i\theta})}{f(e^{i\theta})} e^{i\theta}\, d\theta.$$

In other words, the average value of $zf'(z)/f(z)$ over all z on the unit circle is equal to N. Since N is real, the same is true for $\operatorname{Re}(zf'(z)/f(z))$; and since the maximum of a real-valued function is at least as big as its average, we have

$$\max_{z\in \partial D} \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \# \text{ of zeros of } f \text{ in } D.$$

Finally, since $\operatorname{Re}(zf'(z)/f(z))$ is a harmonic function, its maximum value occurs on the boundary. □

3. Evaluate

$$\int_0^\infty \frac{\cos ax}{\cosh x}\,dx$$

where a is a positive real constant.

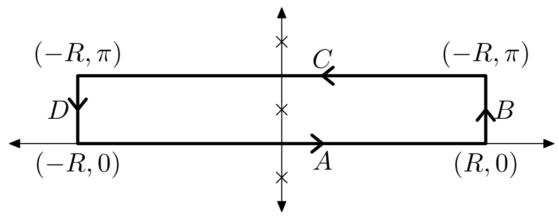
Answer. The integrand is even and real, so we have

$$\int_0^\infty \frac{\cos ax}{\cosh x}\,dx = \frac{1}{2}\,\mathrm{Re}\left(\int_{-\infty}^\infty \frac{e^{iax}}{\cosh x}\,dx\right)$$

Considered as a complex function, the integrand

$$f(z) = \frac{e^{iaz}}{\cosh z}$$

has a pole at $(n + 1/2)i\pi$ for each integers n. Consider the integral of $f(z)$ around a rectangle that goes from $-R$ to R along the real axis and 0 to π along the imaginary axis, as shown below.



On one hand, the integral includes one pole of $f(z)$ at $z = i\pi/2$. We have

$$\frac{1}{\cosh(z + i\pi/2)} = \frac{2}{e^{z+i\pi/2} + e^{z-i\pi/2}} = \frac{2}{ie^z - ie^{-z}} = -i\frac{1}{z + z^3/3! + \cdots} = -i\frac{1}{z} + \cdots$$

so the residue of $1/\cosh z$ at $z = i\pi/2$ is $-i$ and the residue of $f(z)$ is $-ie^{-a\pi/2}$.

On the other hand, we have

$$f(z + i\pi) = \frac{e^{ia(z+i\pi)}}{\cosh(z + i\pi)} = e^{-a\pi}\frac{e^{iaz}}{\cosh z} = e^{-a\pi} f(z).$$

Therefore,

$$\begin{aligned}\oint_{A+B+C+D} f(z)\,dz &= 2\pi e^{-a\pi/2} \\ &= \int_{B+D} f(z)\,dz + \int_{-R}^{R} (f(z) - f(z + i\pi))\,dz \\ &= \int_{B+D} f(z)\,dz + (1 - e^{-a\pi}) \int_{-R}^{R} f(z)\,dz.\end{aligned}$$

Now take the limit as $R \to \infty$. To check that the first term goes to 0, note that $\cos az$ is periodic with period $2\pi/a$ and so is bounded as we move the B and D intervals outward, while $lim_{x\to\pm\infty} 1/\cosh x = 0$. Since the length of the B and D intervals is constant, this implies that the first term goes to zero. We therefore have

$$\begin{aligned}
(1 - e^{a\pi}) \int_{-\infty}^{\infty} f(z)\,dz &= 2\pi e^{-a\pi/2} \\
\int_{-\infty}^{\infty} f(z)\,dz &= \frac{2\pi}{e^{a\pi/2} - e^{-a\pi/2}} = \frac{\pi}{\sinh a\pi} \\
\int_0^{\infty} \frac{\cos ax}{\cosh x}\,dx &= \frac{\pi}{2\sinh a\pi}.n
\end{aligned}$$

You can also do this integral by integrating around a larger rectangle (from 0 to $N\pi$ along the imaginary axis) that includes several poles. If you do it that way, you end up with a sum of a geometric series. □

4. Let $f : D \to \mathbb{C}$ be a function on the unit disk which satisfies $|f(z)| < 1$. Show that if $f(\alpha) = 0$, then

$$|f(z)| \leq \left| \frac{z - \alpha}{\bar{\alpha}z - 1} \right|.$$

Answer. Recall that

$$\frac{z - \alpha}{\bar{\alpha}z - 1}$$

is a Möbius transformation that maps the unit disk to itself and takes α to 0. Define

$$g(z) = f\left(\frac{z - \alpha}{\bar{\alpha}z - 1}\right).$$

Then g is an analytic function mapping D to D, and $g(0) = 0$. Thus

$$|f(z)| = |g\left(\frac{z - \alpha}{\bar{\alpha}z - 1}\right) \leq \left| \frac{z - \alpha}{\bar{\alpha}z - 1} \right|.$$

by Schwarz' Lemma applied to $g(z)$. □

5. A function $f : \mathbb{C} \to \mathbb{C}$ is said to be *quasiperiodic* with period ω if there are constants $a, b \in \mathbb{C}$ so that

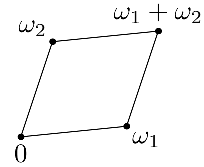
$$f(z + \omega) = e^{az+b} f(z).$$

In this problem we will consider doubly quasiperiodic functions, with two periods ω_1 and ω_2 (with $\operatorname{Im} \omega_2/\omega_1 > 0$, so that ω_1 and ω_2 span a lattice), satisfying

$$\begin{aligned}
f(z + \omega_1) &= e^{a_1 z + b_1} f(z) \\
f(z + \omega_2) &= e^{a_2 z + b_2} f(z).
\end{aligned}$$

(a) Find all entire, doubly quasiperiodic functions which are never zero.

Answer. Let M be the maximum value of $f(z)$ inside the closed parallelogram P with sides ω_1 and ω_2 as below.



The value of $f(z)$ at any point $z \in \mathbb{C}$ can be related to $f(z_0)$ for some $z_0 \in P$ by repeatedly applying the two relations above. The number of times we need to apply the relations is $O(|z|)$, and each time we apply the relation we pick up a factor of $O(e^{|z|})$, so in all the value of $f(z)$ is $O(e^{|z|^2})$. We proved that any function which grows as $O(e^{|z|^\alpha})$ and has no zeros can be written as $e^{Q(z)}$ for some polynomial $Q(z)$ with degree less than or equal to α; in this case, $f(z)$ must be of the form

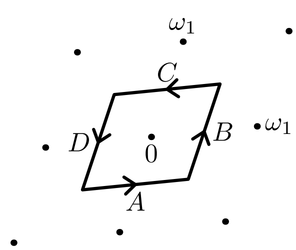
$$f(z) = e^{pz^2+qz+r}$$

for constants p, q, and r. It is straightforward to check that every such function is quasiperiodic. □

(b) If $f(z)$ is entire and doubly quasiperiodic and has a simple zero at $n_1\omega_1 + n_2\omega_2$ for all integers n_1, n_2 and no other zeros, show that

$$a_1\omega_2 - a_2\omega_1 = 2\pi i.$$

Answer. Consider the integral around the parallelogram show below.



(Note that the orientations are as shown because $\text{Im}(\omega_2/\omega_1) > 0$.) Then, since the path contains just one zero, we have

$$\oint_{A+B+C+D} \frac{f'(z)}{f(z)}\, dz = 2\pi i.$$

We also have

$$\frac{f'(z+\omega_2)}{f(z+\omega_2)} = \frac{e^{a_2 z+b}(a_2 f(z) + f'(z))}{e^{a_2 z+b} f(z)}$$

$$= a_2 + \frac{f'(z)}{f(z)}$$

$$\int_C \frac{f'(z)}{f(z)}\, dz = \int_{-A} \left(a_2 + \frac{f'(z)}{f(z)} \right) dz$$

$$= -a_2\omega_1 - \int_A \frac{f'(z)}{f(z)}\, dz$$

Similarly,

$$\int_B \frac{f'(z)}{f(z)}\, dz = a_1\omega_2 - \int_D \frac{f'(z)}{f(z)}\, dz$$

and so the total integral reduces to

$$\oint_{A+B+C+D} \frac{f'(z)}{f(z)}\, dz = a_1\omega_2 - a_2\omega_1$$

as desired. □

6. Let $h : D \setminus \{0\} \to \mathbb{R}$ be a harmonic function on the punctured unit disk. Suppose that

$$\lim_{z\to 0} zh(z) = 0.$$

Show that

$$h(z) = \alpha \log(|z|) + g(z)$$

for a suitable $\alpha \in \mathbb{R}$ and harmonic function $g : D \to \mathbb{R}$ defined on the whole unit disk (including $z = 0$).

Answer. Define $H(z) = h(e^z)$. $H(z)$ is a harmonic function on the left half plane, which is periodic with period $2\pi i$. Since it is a harmonic function on a simply connected set, it is the real part of an analytic function $f(z)$ defined on the left half plane. The real part of $f(z)$ is periodic, which means that

$$F(z + 2\pi i) = F(z) + A(z)$$

where $A(z)$ is purely imaginary and analytic. Therefore, $A(z)$ is a constant, say $2\pi i\alpha$. Now define $G(z) = F(z) - \alpha z$; note that $G(z)$ is periodic with period $2\pi i$. Therefore, we can define an analytic function $\tilde{g}(z)$ on the unit disk by $\tilde{g}(z) = G(\log z)$. Set $g(z) = \operatorname{Re} \tilde{g}(z)$. Note that

$$g(z) = h(z) - \alpha \log(|z|)$$

and so

$$\lim_{z\to 0} zg(z) = \lim_{z\to 0} zh(z) = 0$$

which implies that $\tilde{g}(z)$ has a removable singularity at 0: If $\tilde{g}(z)$ had an essential singularity at 0, then the values of $\tilde{g}(z)$ would be dense in $\mathbb{C}$ in any neighborhood of 0 and so the values of $g(z)$ would also be dense in $\mathbb{R}$. If $\tilde{g}(z)$ had a pole at 0, then $\tilde{g}(z)$ takes all values in a neighborhood of ∞ in any neighborhood of 0, so again the values of $g(z)$ in any neighborhood of 0 are dense in $\mathbb{R}$.

Thus $\tilde{g}(z)$ can be extended to an analytic function defined on the whole unit disk, and thus $g(z)$ can be extended to a harmonic function on the whole disk, as desired. $\square$