MATH 113 PROBLEM SET 5, PARTIAL SOLUTIONS

Due October 17, 2003

(5) *In class, we saw that, for a polynomial $p(z)$ and a simple closed curve Γ, the number of zeros of p inside Γ is equal to the winding number of $p(\Gamma)$ around 0.*

(a) *Show that this is also true for a general analytic function $f(z)$ which is analytic on a disk containing Γ.*

Approach 1: "Factor out" the zeros of $f(z)$ to get another function $g(z)$ and show that the winding number of $g(\Gamma)$ around 0 is equal to 0.

Suppose that $f(z)$ is analytic on the disk D. The set of zeros of $f(z)$ has no accumulation points, so $f(z)$ has only a finite number of zeros on D. Let $\alpha_1, \ldots, \alpha_n$ be the zeros (with multiplicities) of $f(z)$. Then the function

$$g(z) = \frac{f(z)}{(z-\alpha_1)\cdots(z-\alpha_n)}$$

is analytic away from the α_i, can be extended continuously to the α_i, and so can be extended to an analytic function on D; furthermore, $g(z)$ is non-zero. Then, as in the case of polynomials, we have that the winding number of a product equals the sum of the winding numbers:

$$\begin{aligned} n(f(\Gamma), 0) &= n(\Gamma - \alpha_1, 0) + \cdots + n(\Gamma - \alpha_n, 0) + n(g(\Gamma), 0) \\ &= n(\Gamma, \alpha_1) + \cdots + n(\Gamma, \alpha_n) + n(g(\Gamma), 0) \end{aligned}$$

Each term $n(\Gamma, \alpha_i)$ is 1 if α_i is contained inside the simple curve Γ, and 0 otherwise, so the first terms give the total number of zeros of $f(z)$ contained inside Γ. It remains to show that the last term, $n(g(\Gamma), 0)$, is 0. Since the disk D is simply connected, there is a homotopy from Γ to a trivial loop inside of D. (See the handout for an explicit homotopy.) Composing with g, we also get a homotopy from $g(\Gamma)$ to a trivial loop; since g is never 0, this is a homotopy inside of $\mathbb{C} \setminus \{0\}$, so the winding number does not change over the course of the homotopy, and $n(g(\Gamma), 0) = 0$.

Approach 2: Show that

$$\int_{f(\Gamma)} \frac{dz}{z} = \int_\Gamma \frac{f'(z)}{f(z)} dz$$

and analyze the behaviour of $f'(z)/f(z)$ near a zero of $f(z)$.

Let the curve Γ be parametrized by $z = \gamma(t)$, with $t \in [0, 1]$. Then, by definition of the complex line integral,

$$\int_\Gamma \frac{f'(z)}{f(z)} dz = \int_0^1 \frac{f'(\gamma(t))}{f(\gamma(t))} \gamma'(t)\, dt,$$

while

$$\int_{f(\Gamma)} \frac{dz}{z} = \int_0^1 \frac{\frac{d}{dt} f(\gamma(t))}{f(\gamma(t))} dt$$
$$= \int_0^1 \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t))} dt$$

These two are the same, so to find the winding number of $f(\Gamma)$ around 0 (given by $\frac{1}{2\pi i}\int_{f(\Gamma)} dz/z$), it suffices to evaluate the integral $\frac{1}{2\pi i}\int_\Gamma f'(z)/f(z)\,dz$. The easiest way to calculate this integral is to apply the Residue Theorem. The poles of $f'(z)/f(z)$ can only occur when the denominator $f(z)$ has a zero. Suppose the denominator $f(z)$ has a zero of order k at α; then we have a power series expansion

$$f(z) = a_k(z-\alpha)^k + a_{k+1}(z-\alpha)^{k+1} + \cdots$$
$$f'(z) = ka_k(z-\alpha)^{k-1} + (k+1)a_{k+1}(z-\alpha)^k + \cdots$$
$$\frac{f'(z)}{f(z)} = k(z-\alpha)^{-1} + \cdots$$

so $f'(z)/f(z)$ has a pole with residue k at α. Now $\frac{1}{2\pi i}\int_\Gamma f'(z)/f(z)\,dz$ is the sum of the residues of the poles of $f'(z)/f(z)$ at the poles included inside Γ (since Γ is a simple closed curve, so all the winding numbers are 1), which, by the computation above, is the sum of the orders of the zeros of $f(z)$ inside of Γ, as desired.

(b) *Define the "inside" of a general curve Γ to be the set of points $z \in \mathbb{C}$ so that $n(\Gamma, z) \neq 0$. Show that, under the assumptions above, every point in the inside of Γ gets mapped to the inside of $f(\Gamma)$.*

(Recall that the assumptions above were that f is analytic on a disk and that Γ is a simple closed curve contained inside the disk.)
First let us generalize the result from the first part. We found that the number of zeros of $f(z)$ inside of Γ was equal to the winding number of $f(\Gamma)$ around 0. By replacing $f(z)$ by $f(z) - a$, we can also see that the number of times f takes the value a inside Γ (with multiplicity) equals the winding number of $f(\Gamma)$ around a. In particular, for any point z on the inside of Γ, the value $f(z)$ is obtained at least once, so the winding number of $f(\Gamma)$ around $f(z)$ is at least one, and so $f(z)$ is on the inside of $f(\Gamma)$ as defined in the statement of the problem.
Note that it is crucial that Γ is a simple closed curve: if not, we would have to take into account the winding number of Γ around the different points at which f takes the value a; if Γ has negative winding number around some points, we could get cancellation.

(c) *Show how this result generalizes the Maximum Modulus principle.*

Let R be the maximum value of $|z|$ on the curve $f(\Gamma)$. Then every point on the inside of $f(\Gamma)$ is contained inside the disk D_R of radius R around 0. There are several ways to see this. For instance, for a outside of D_R, a straight line from a out to infinity does not intersect $f(\Gamma)$, so the winding number is 0 according to the counting intersections criterion. But by the previous part, every point on the inside of Γ maps to the inside of $f(\Gamma)$, and so in particular maps inside D_R, and so the maximum absolute value of the function f on the region bounded by Γ occurs on Γ itself.

(7) (Note: there were several typos in the statement of this problem; in particular, the behavior of $\epsilon(x, y)$ as $x, y \to 0$ was stated wrong. A corrected statement is below.) *Let $f(z)$ be an*

arbitrary function on $\mathbb{C}$ with values in $\mathbb{C}$ with two continuous (real) derivatives. In other words, near each $z_0 \in \mathbb{C}$, f can be written as

$$f(z_0 + x + iy) = f(z_0) + f_x(z_0)x + f_y(z_0)y + f_{xx}(z_0)\frac{x^2}{2} + f_{xy}(z_0)xy + f_{yy}(z_0)\frac{y^2}{2} + \epsilon(x, y)$$

where $\lim_{x,y\to 0} \epsilon(x, y)/|x + iy|^2 = 0$.

Define the Laplacian *of f as*

$$\Delta f(z) = \left(\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2\right) f(z) = f_{xx}(z) + f_{yy}(z).$$

(a) *Show that*

$$\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta = f(z_0) + \frac{1}{4}r^2\Delta f(z_0) + \epsilon'(r)$$

where $\lim_{r\to 0} \epsilon'(r)/r^2 = 0$.

By the Taylor expansion for $f(z_0 + re^{i\theta})$, we can rewrite the integral, with $x = r\cos\theta$ and $y = r\sin\theta$:

$$\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi} \big(f(z_0) + f_x(z_0)r\cos\theta + f_y(z_0)r\sin\theta +$$
$$+ f_{xx}(z_0)\frac{r^2}{2}\cos^2\theta + f_{xy}(z_0)r^2\sin\theta\cos\theta + f_{yy}(z_0)\frac{r^2}{2}\sin^2\theta + \epsilon(r\cos\theta, r\sin\theta)\big)d\theta$$

All the trigonometric integrals are easy to do. Three of them are zero, so we have:

$$\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta = f(z_0) + \frac{r^2}{4}(f_{xx}(z_0) + f_{yy}(z_0)) + r^2\int_0^{2\pi} \epsilon(r\cos\theta, r\sin\theta)d\theta$$

If we set $\epsilon'(r) = r^2\int_0^{2\pi} \epsilon(r\cos\theta, r\sin\theta)d\theta$, we have the desired expansion. Finally, we need to check that we have a valid Taylor expansion:

$$\lim_{r\to 0} \epsilon'(r)/r^2 = \lim_{r\to 0}\int_0^{2\pi} (\epsilon(r\cos\theta, r\sin\theta)/r^2)d\theta$$

But the integrand is just $\epsilon(x, y)/|x+iy|^2$ evaluated at $x = r\cos\theta$ and $y = r\sin\theta$, which goes to 0 as r goes to 0.

(b) *Show that if f is harmonic, then Δf is identically 0.*

If f is harmonic, then the average value on any circle centered at z_0 equals the value at the center:

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta \\ &= f(z_0) + \frac{r^2}{4}\Delta f(z_0) + \epsilon'(r). \end{aligned}$$

Subtracting $f(z_0)$ from both sides and dividing by r^2, we find:

$$0 = \frac{1}{4}\Delta f(z_0) + \frac{\epsilon'(r)}{r^2}$$

Taking the limit as $r \to 0$, we get the desired result.

(c) *Conversely, use the factorization*

$$\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

to show that if $\Delta f = 0$*, then there exist analytic functions* $g(z)$ *and* $h(z)$ *so that*

$$f(z) = g(z) + \bar{h}(z).$$

That is, the only harmonic functions are sums of analytic and anti-analytic functions.
Consider the function

$$p(z) = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f(z).$$

Since f satisfies $\Delta f = 0$, we have

$$\Delta f(z) = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) p(z) = 0.$$

But this says that

$$p_x(z) + ip_y(z) = 0$$

or, in other words, that g satisfies the Cauchy-Riemann equations $p_y = ip_x$. Since the second derivatives of p exist, p_x and p_y are continuous, and so $p(z)$ is complex-differentiable and therefore analytic.

Now suppose that we had

$$f(z) = g(z) + \bar{h}(z)$$

as desired. Then we would have

$$p(z) = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f(z) = (g_x - ig_y)(z) + (\bar{h}_x - i\bar{h}_y)(z).$$

But if g is analytic, then $g_y(z) = ig_x(z)$ and $g_x(z) = g'(z)$, while if $\bar{h}$ is anti-analytic, $\bar{h}_y(z) = -i\bar{h}_x(z)$, so this becomes

$$p(z) = 2g_x(z) = 2g'(z).$$

To return to what we actually have, this suggests that we define g in terms of p:

$$g(z) = \frac{1}{2}\int_0^z p(w)\, dw.$$

Now $\bar{h}(z)$ is determined: define

$$\bar{h}(z) = f(z) - g(z).$$

Let's check that $\bar{h}$ is anti-analytic:

$$\begin{aligned}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\bar{h}(z) &= \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f(z) - \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) g(z) \\ &= p(z) - 2g'(z) \\ &= 0.\end{aligned}$$

We used twice the fact, which is easy to check, that a function is anti-analytic if and only if it has continuous first partial derivatives and satisfies the anti-Cauchy-Riemann equations:

$$\bar{h}_y(z) = -i\bar{h}_x(x).$$