Solutions for Problem Set #3
due October 3, 2003
Dustin Cartwright

1 *Show that your favorite trigonometric identity continues to hold for complex values of the arguments. (You might, for instance, show that the usual angle-addition formulas for* $\sin(z+w)$ *and* $\cos(z+w)$ *continue to hold for complex values of* z *and* w*.) (10 points)*

I'll show the $\sin^2$ formula:

$$\begin{aligned}\sin^2 z &= \left(\frac{e^{zi} - e^{-zi}}{2i}\right)^2 \\ &= \frac{e^{2zi} - 2e^0 + e^{-2zi}}{-4} \\ &= \frac{1}{2} - \frac{e^{2zi} + e^{-2zi}}{4} \\ &= \frac{1 - \cos 2z}{2}\end{aligned}$$

Note that all of the trigonometric identities can be proved at once using the rigidity theorem: Both sides of the equation are entire functions (if there are two or more variables fix one), which are equal on the real line. Therefore, they must be equal on the entire complex plane.

2a. (B&N 3.11a.) *Show that* e^z *is entire by verifiying the Cauchy-Riemann equations for its real and imaginary parts. (not graded)*

$$\begin{aligned}u(x,y) &= e^x \cos y \\ v(x,y) &= e^x \sin y \\ \frac{\partial u}{\partial x} &= e^x \cos y \\ \frac{\partial u}{\partial y} &= -e^x \sin y \\ \frac{\partial v}{\partial x} &= e^x \sin y \\ \frac{\partial v}{\partial y} &= e^x \cos y\end{aligned}$$

so e^z satisfies the Cauchy-Riemann equations and is analytic.

b. *Prove* $e^{z_1+z_2} = e^{z_1}e^{z_2}$ *(not graded)*

$$\begin{aligned} e^{(x_1+iy_1)+(x_2+iy_2)} &= e^{x_1+x_2}(\cos(y_1+y_2)+i\sin(y_1+y_2)) \\ &= e^{x_1}e^{x_2}\big(\cos(y_1)\cos(y_2)-\sin(y_1)\sin(y_2) \\ &\quad +i\big(\sin(y_1)\cos(y_2)+\cos(y_1)\sin(y_2)\big)\big) \\ &= e^{x_1}e^{x_2}(\cos(y_1)+i\sin(y_1))(\cos(y_2)+i\sin(y_2)) \\ &= e^{x_1+iy_1}e^{x_2+iy_2} \end{aligned}$$

3 (B&N 3.17) *Find* $\sin^{-1}(2)$, *that is, find solutions of* $\sin z = 2$. *(10 points)*

Using the definition of sin:

$$\begin{aligned} \frac{e^{iz}-e^{-iz}}{2i} &= 2 \\ e^{iz}-e^{-iz} &= 4i \end{aligned}$$

Now set $w = e^{ix}$, so that this becomes:

$$\begin{aligned} w - w^{-1} &= 4i \\ w^2 - 4iw - 1 &= 0 \end{aligned}$$

By the quadratic formula,

$$\begin{aligned} w &= \frac{4i \pm \sqrt{-16+4}}{2} \\ &= 2i \pm \sqrt{3}i \end{aligned}$$

Therefore, we can solve for z,

$$\begin{aligned} iz &= \ln(2\pm\sqrt{3}) + (\frac{\pi}{2} + 2k\pi)i \\ z &= -i\ln(2\pm\sqrt{3}) + \frac{\pi}{2} + 2k\pi \end{aligned}$$

for any integer value of k.

Corina Patrascu came up with the following alternate solution:

$$\cos z = \pm\sqrt{1 - \sin^2 z} = \pm\sqrt{3}i$$

Therefore, by Euler's formula,

$$e^{iz} = \cos z + i \sin z = \pm\sqrt{3}i + 2i$$

and then we solve for z as above.

4 (B&N 3.18) *Show that* $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$. *(not graded)*

$$\begin{aligned}
\sin(x + iy) &= \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} \\
&= \frac{e^{ix-y} - e^{-ix+y}}{2i} \\
&= \frac{e^{-y}(\cos x + i \sin x) - e^{y}(\cos x - i \sin x)}{2i} \\
&= i \sin x \frac{e^{-y} + e^{y}}{2i} + \cos x \frac{e^{-y} - e^{y}}{2i} \\
&= \sin x \frac{e^{y} + e^{-y}}{2} + i \cos x \frac{e^{y} - e^{-y}}{2} \\
&= \sin \cosh y + i \cos x \sinh y
\end{aligned}$$

5 (Needham 2.12a.) *Consider the geometric series* $P(z) = \sum_{j=0}^{\infty} z^j$, *which converges to* $1/(1 - z)$ *inside the unit disk. The approximating-polynomials in this case are* $P_m(z) = \sum_{j=0}^{m} z^j$. *(not graded)*

(i) *Show that the error* $E_m(z) \equiv |P(z) - P_m(z)|$ *is given by*

$$E_m(z) = \frac{|z^{m+1}|}{|1 - z|}$$

$$
\begin{aligned}
|P(z) - P_m(z)| &= \left| \sum_{j=m+1}^{\infty} z^j \right| \\
&= \left| z^{m+1} \sum_{j=0}^{\infty} z^j \right| \\
&= |z|^{m+1} \left| \frac{1}{1-z} \right|
\end{aligned}
$$

(ii) *If z is any fixed point in the disc of convergence, what happens to the error as m tends to infinity.*

The error goes to zero because $|z| < 1$.

(iii) *If we fix m, what happens to the error as z approaches the boundary point $z = 1$?*

The error goes to infinity because the denominator, $|1-z|$ goes to 0, and the numerator remains positive.

(iv) *Suppose we want to approximate this series in the disc $|z| \leq 0.9$, and further suppose that the maximum error we will tolerate is $\epsilon = 0.01$. Find the lowest degree polynomial $P_m(z)$ that approximates $P(z)$ with the desired accuracy throughout the disc.*

$$
\begin{aligned}
\frac{|z^{m+1}|}{|1-z|} &\leq \epsilon \\
|z^{m+1}| &\leq 0.1 \cdot 0.01 \\
&\leq |1-z|\epsilon \\
m+1 &\geq \log_{0.9} 0.001 \\
&\geq \log_z 0.001 \\
m+1 &\geq 65.56
\end{aligned}
$$

Therefore, the lowest degree polynomial is $m = 65$.

6 (Needham 2.15) *Give an example of a pair of origin-centered power series, say $P(z)$ and $Q(z)$, such that the disc of convergence for the product $P(z)Q(z)$ is larger than either of the two discs of convergenc for $P(z)$ and $Q(z)$. (10 points)*

We know that $1/(1-z)$ is expressible as a power series, so by simple multiplication, so is $\frac{1+z}{1-z}$. By substituting $-z$ for z, we get a power series for $\frac{1-z}{1+z}$. Both of these have radius of convergence no more than 1, because they each have a singularity on the unit disc, at 1 and -1 respectively. However, their product is the constant function 1, which is itself a power series which converges everywhere.

7 *Consider the mapping $f(z) = z^2$.*

7 i *Find the images of horizontal and vertical lines under this mapping. (10 points)*

In terms of its real and imaginary parts,

$$f(x+iy) = x^2 + 2ixy - y^2 = (x^2 - y^2) + 2ixy$$

Let $a + bi$ be the point in the image, i.e. $a = x^2 - y^2$, and $b = 2xy$. If we hold y constant, then we get the image of a horizontal line, which can be described by the equation $a = (b/2y)^2 - y^2$, which is a parabola facing in the positive real direction. If we hold x constant, we get the image of a vertical line, which can be described by the equation $a = x^2 - (b/2x)^2$, which is a parabola open towards the negative real direction.

7 ii *Show directly that the image curves intersect at right angles and thus that $f(z)$ maps little squares to little rectangles. (10 points)*

The derivative of the image of the horizontal line at the image of $x + iy$ is:

$$\begin{aligned}\frac{\partial a}{\partial b} &= \frac{\partial}{\partial b}\left(\frac{b}{2y}\right)^2 - y^2 \\ &= \frac{b}{2y^2} \\ &= \frac{2xy}{2y^2} \\ &= \frac{x}{y}\end{aligned}$$

For the image of the vertical line:

$$\begin{aligned}\frac{\partial a}{\partial b} &= \frac{\partial}{\partial b}x^2 - \left(\frac{b}{2x}\right)^2 \\ &= -\frac{b}{2x^2} \\ &= -\frac{2xy}{2x^2} \\ &= -\frac{y}{x}\end{aligned}$$

Therefore, the partials multiply to -1, which is the condition for two slopes to be perpendicular.

7 iii *Find geometrically the image of an arbitrary line. Show that the map is conformal when $z \neq 0$ by showing that the angle between two arbitrary lines is preserved by the mapping. (5 points extra credit)*

An arbitrary line can be parametrized by a real variable t as $at + b$, where $a \neq 0$ and b are complex numbers. Rewriting this as $a(t + b/a)$, then $(a(t + b/a))^2 = a^2 f(t + b/a)$. Therefore, $f(t + b/a)$ is just the image of a horizontal line, which is a parabola, and then multiplying by a^2 just rotates this parabola.

8 *We saw in class that the exponential function maps vertical straight lines to circles centered at the origin. The goal of this problem is to see to what extent the converse is true: what are all the conformal functions which map vertical lines to circles?*

8 i *Consider the map* $F(x, y) = f(x)(\cos y + i \sin y)$*, for a real valued function* $f(x)$*. Show that* F *maps vertical lines to circles. Find all such functions* $f(x)$ *so that* F *is a conformal map. (10 points)*

On vertical lines x is constant, so $f(x)$ is constant, and so $F(x, y)$ just traces out a circle with radius $f(x)$.

The partial derivatives are:

$$\begin{aligned} iF_x &= if'(x)(\cos y + i \sin y) \\ &= f'(-\sin y + i \cos y) \\ F_y &= f(x)(-\sin y + i \cos y) \end{aligned}$$

In order for these two to be equal for all y, $f'(x)$ must equal $f(x)$. However, the solutions to this differential equation are known to just be $f(x) = Ae^x$ for some real constant A.

8 ii *What happens if* $f(x)$ *is complex-valued? Do you get any more solutions? (10 points)*

The Cauchy-Riemann equations remain the same, as does the differential equation. If f is allowed to take on complex values, the only difference is that the constant A may be complex.

8 iii *Extend the argument above to find all complex differentiable functions on* $\mathbb{C}$ *which map vertical lines to circles. (5 points extra credit)*

Such a function must be of the form

$$F(x, y) = f(x)\big(\cos(\theta(x, y)) + i \sin(\theta(x, y))\big)$$

In order for F to be conformal, the images of horizontal lines must be perpendicular to the circles around the origin. This means that they must be radial, so θ cannot depend on y. Now the partial derivatives are:

$$\begin{aligned} iF_x &= f'(x)\big(-\sin(\theta(y)) + i \cos(\theta(y))\big) \\ F_y &= f(x)\big(\cos(\theta(y)) + i \cos(\theta(y))\big)\theta'(y) \end{aligned}$$

Similar to above, we must have that $f'(x) = f(x)\theta'(y)$. Since this must be true for all x and all y, and $f(x)$ is not uniformly zero, $\theta'(y)$ is constant. Call this α. Then, similar to part ii, $f(x) = Ae^{\alpha x}$. Also, $\theta(y) = \alpha y + \beta$. Therefore,

$$\begin{aligned} F(x, y) &= Ae^{\alpha x}e^{\alpha yi+\beta i} \\ &= Ae^{\beta i}e^{\alpha z} \\ &= A'e^{\alpha z} \end{aligned}$$