Math 113 Assignment 5 Solutions

Problem 1 (§2.2, 11). a) We will compute each of these integrals using polar coordinates:

$$\int_{|z|=1} \frac{dz}{z} = \int_{\theta=0}^{2\pi} \frac{d(e^{i\theta})}{e^{i\theta}} = \int_0^{2\pi} \frac{ie^{i\theta}}{e^{i\theta}} d\theta = \int_0^{2\pi} i d\theta = 2\pi i$$

$$\int_{|z|=1} \frac{dz}{|z|} = \int_{\theta=0}^{2\pi} \frac{d(e^{i\theta})}{|e^{i\theta}|} = \int_0^{2\pi} = \int_0^{2\pi} ie^{i\theta} d\theta = i\left[-ie^{i\theta}\right]_0^{2\pi} = 0$$

$$\int_{|z|=1} \frac{|dz|}{z} = \int_{\theta=0}^{2\pi} \frac{|d(e^{i\theta})|}{e^{i\theta}} = \int_0^{2\pi} \frac{|ie^{i\theta}|}{e^{i\theta}} d\theta = \int_0^{2\pi} e^{-i\theta} d\theta = \left[ie^{-i\theta}\right]_0^{2\pi} = 0$$

$$\int_{|z|=1} \left|\frac{dz}{z}\right| = \int_{\theta=0}^{2\pi} \left|\frac{ie^{i\theta}}{e^{i\theta}}\right| d\theta = \int_0^{2\pi} d\theta = 2\pi.$$

b) Since z^2 is an entire function and $\gamma(0) = \gamma(\pi/2) = 0$, we know by theorem 2.1.9 that $\int_\gamma z^2 dz = 0$.

Problem 2 (§2.2, 12). We know that there is a branch of $\log z$ that is analytic on $\mathbb{C} \setminus \mathbb{R}_{\geq 0} \supset \mathbb{C} \setminus \{z \mid \operatorname{Re} z \leq 0\}$. Therefore, there is a global antiderivative for $1/z$ on $\mathbb{C} \setminus \{z \mid \operatorname{Re} z \leq 0\}$, which means that $\int_\gamma (1/z) dz = 0$ by theorem 2.1.9.

Problem 3 (§2.2, 13). Since $z \sin z^2$ is an entire function, $\int_\gamma z \sin z^2 dz = 0$ by theorem 2.1.9.

Problem 4 (§2.2, 14). If γ is homotopic to a union of simple closed curves, none of which contains the origin, then $\int_\gamma (1/z) dz = 0$. One way to see this is to break up a curve whenever it crosses the negative real axis, introducing boundaries that cancel each other out as in figure 2.2.4, to write γ as a union of closed curves that are each contained in the principal domain of $\log z$. Alternatively, we can justify this condition topologically by noting that $e^z : \mathbb{C} \to \mathbb{C}^*$ is a covering map, and that the condition is enough to guarantee that every lift of γ from $\mathbb{C}^*$ to $\mathbb{C}$ is a closed curve.

Problem 5 (§2.2, 5). We want to know when

$$\int_\gamma \frac{dz}{z^2 + z + 1} = 0$$

for closed simple curves γ. Note that this has singularities at $z_1, z_2 = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ and is analytic elsewhere in the plane. Before continuing, we compute the partial fractions expansion. I omit the algebra, but a similar computation can be seen in the previous problem set.

$$\frac{1}{z^2 + z + 1} = \frac{-i/\sqrt{3}}{z - \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)} + \frac{i/\sqrt{3}}{z - \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)}.$$

If γ does not enclose any singularities (in the language of winding numbers, $I(\gamma, z_1) = 0$, $I(\gamma, z_2) = 0$). Then by Cauchy's theorem the integral around γ is 0.

Suppose that γ encloses exactly one of the singularities. Let's just suppose γ encloses $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ but not the other singularity. Then γ is homotopic to the unit circle around $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ (this circle does not enclose the other root). Parametrize this by $\gamma(t) = e^{it} - \frac{1}{2} + i\frac{\sqrt{3}}{2}$.

Then

$$\begin{aligned}\int_\gamma \frac{dz}{z^2+z+1} &= \int_\gamma \left(\frac{-i/\sqrt{3}}{z - \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)} + \frac{i/\sqrt{3}}{z - \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)} \right) dz \\ &= \int_0^{2\pi} \frac{(-i/\sqrt{3}) i e^{it}}{e^{it}}\, dt + 0 = \frac{2\pi}{\sqrt{3}}\end{aligned}$$

The first integral is 0 since this function as an antiderivative which is This is nonzero, so the integral around γ encircling only this root is not 0. A similar computation shows that the integral around a simple closed curve containing only the other singularity is $-\frac{2\pi}{\sqrt{3}}$.

At last, consider a curve which encloses both singularities. This curve is homotopic to a path which is follows the first circle above, then goes down along the imaginary axis, and then around the other circle, and back up the axis. (I'm not feeling ambitious using xypic right now, so you'll have to imagine this. It looks like two circles with a segment between them). The integral over this path is the sum of the integrals over the two circles, since the integrals over the two segments cancel each other out because they're in opposite directions). Then by the preceding computations, the integral is 0!

So $\int_\gamma f(z)\, dz = 0$ if γ encloses neither root or both of them.

Problem 6 (§2.2, 6). Consider the branch of logarithm defined on $\mathbb{C}$ without the negative real axis and 0 and with $\arg z \in [-\pi, \pi)$. The function $F(z) = z^2/2 - \log z$ is analytic on this set and is an antiderivative of $z - 1/z$ there. Then by the fundamental theorem of calculus,

$$\int_\gamma \left(z - \frac{1}{z}\right) dz = \left(\frac{z^2}{2} - \log z\right)\Bigg|_1^i = \left(-\frac{1}{2} - \log i\right) - \left(\frac{1}{2} - \log 1\right) = -1 - \frac{\pi}{2}i.$$

Problem 7 (§2.2, 7). No, this does not hold in general. Let $f(z) = z$ and integrate around the unit circle $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$. By Cauchy's theorem, $\int_\gamma f\, dz = 0$, but for the real part,

$$\int_\gamma \operatorname{Re} f\, dz = \int_0^{2\pi} (\cos t)(ie^{it})\, dt = \int_0^{2\pi} -\sin t \cos t\, dt + i \int_0^{2\pi} \cos^2 t\, dt = 0 + i\pi = i\pi \neq 0,$$

so Cauchy's theorem doesn't hold. Similarly, for the imaginary part,

$$\int_\gamma \operatorname{Im} f\, dz = \int_0^{2\pi} (\sin t)(ie^{it})\, dt = \int_0^{2\pi} -\sin^2\, dt + i \int_0^{2\pi} \sin t \cos t\, dt = -\pi + 0 = -pi \neq 0.$$

Cauchy's theorem doesn't hold for the imaginary part either.

Problem 8 (§2.2, 11). We wish to evaluate

$$\int_\gamma \frac{2z^2 - 15z + 30}{z^3 - 10z^2 + 32z - 32}\, dz$$

around the path γ given by $|z| = 3$. The first step is to expand this in partial fractions. We need a factorization of $z^3 - 10z^2 + 32z - 32$ to do this. We're given that one root is 2, and after polynomial long division we obtain

$$z^3 - 10z^2 + 32z - 32 = (z - 2)(z^2 - 8z + 16) = (z - 2)(z - 4)^2.$$

After some messy algebra which I omit, we can write

$$\int_\gamma \frac{2z^2 - 15z + 30}{z^3 - 10z^2 + 32z - 32}\, dz = \int_\gamma \left(\frac{2}{z - 2} + \frac{1}{(z - 4)^4} \right) dz = \int_\gamma \frac{2}{z - 2}\, dz + \int_\gamma \frac{1}{(z - 4)^2}\, dz.$$

Now, $1/(z - 4)^2$ is anaytic on and inside γ, so the integral around γ is 0. We need only integrate

$$\int_\gamma \frac{2}{z - 2}\, dz.$$

But this curve is homotopic to a circle γ_2 of radius 1 centered at 2, and we've computed $\int_{\gamma_2} 1/(z - 2)\, dz$ many times; it's just $2\pi i$. Thus

$$\int_\gamma \frac{2z^2 - 15z + 30}{z^3 - 10z^2 + 32z - 32}\, dz = 4\pi i.$$