Problem 1 (§3.1, 11). If $|z| > 1$, then

$$\sum_{n=1}^{\infty} \frac{1}{|z^n|} = \frac{1}{1 - 1/|z|} < \infty.$$

In addition, given any compact set $K \subset A$, there exists z_0 for which $|z| \geq |z_0|$ for all $z \in A$. By point-wise convergence, given $\epsilon > 0$ there exists $N > 0$ for which

$$\sum_{n=N}^{\infty} \frac{1}{|z_0^n|} < \epsilon;$$

thus,

$$\sum_{n=N}^{\infty} \frac{1}{|z^n|} \leq \sum_{n=N}^{\infty} \frac{1}{|z_0^n|} < \epsilon$$

for all $z \in K$. We conclude that $\sum 1/z^n$ converges uniformly on compact subsets of A; thus, by the analytic convergence theorem, $\sum 1/z^n$ is analytic on A.

Problem 2 (§3.1, 12). Given $z \in \mathbb{C}^*$, we can see by the ratio test that $\sum 1/(n!z^n)$ converges:

$$\lim_{n\to\infty} \frac{1/(n+1)!}{1/n!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$$

As before, given a compact set $K \subset \mathbb{C}^*$, there exists $z_0 \in K$ for which $z \in K \Rightarrow |z| \geq |z_0|$. This fact, together with point-wise convergence, means that for any $\epsilon > 0$, there exists $N > 0$ with

$$\sum_{n=N}^{\infty} \frac{1}{|n!z^n|} \leq \sum_{n=N}^{\infty} \frac{1}{|n!z_0^n|} < \epsilon.$$

Therefore, $\sum 1/(n!z^n)$ converges absolutely and uniformly on compact subsets of $\mathbb{C}^*$, which implies by the analytic convergence theorem that $\sum 1/(n!z^n)$ is analytic on $\mathbb{C}^*$.

Problem 3 (§3.1,15). By worked example 3.1.15, $\zeta(n) = \sum n^{-z}$ and $\zeta'(n) = \sum -(\log n)n^{-z}$ are both analytic on $A := \{z \mid \operatorname{Re} z > 1\}$. Therefore, $\zeta^{(k)}(z) = (-1)^k(\log n)^k n^{-z}$ is also analytic on A.

Problem 4 (§3.1, 17). It is not always true that $f_n'(z) \to f'(z)$ uniformly. Let A be the unit disk Δ, and let $f_n(z) = \sum_{k=1}^{n} \frac{z^{k+1}}{k(k+1)}$. By the Weierstrauss M-test, with $M_k = 1/(k^2 + k)$ (such that the M_k converge by comparison with the p-series $\sum 1/k^2$), $f_n \to f$ uniformly on Δ. However, Marsden proves on page 190 that the $f_n'(z) = \sum_{k=1}^{n} \frac{z}{k}$ do not converge uniformly on Δ.

Problem 5 (§3.2, 2). We compute each radius of convergence using the ratio test:

a)

$$R = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$$

b)

$$R = \lim_{n\to\infty} \frac{4^{-n/2}}{4^{-(n+1)/2}} = 4^{1/2} = 2$$

c)

$$R = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

d)

$$R = \lim_{n\to\infty} \frac{1/(1+2^n)}{1/(1+2^{n+1})} = \lim_{n\to\infty} \frac{2^{-n}}{2^{-n-1}} = 2$$

Problem 6 (§3.2, 12). Although e^z is entire, we know by the fact that $\lim_{z\to\infty} |e^z| = \infty$ that it is not analytic at $z = \infty$. Therefore, $e^{1/z}$ is not analytic at zero, and the described series does not converge.

Problem 7 (§3.2, 15). By the triangle inequality, $|\operatorname{Re} a_n| \leq |a_n|$ for each n. If $|z| < R$, then by definition of the radius of convergence $\sum |a_n z^n| < \infty$. Therefore,

$$\sum_{n=0}^{\infty} |\operatorname{Re} a_n z^n| \leq \sum_{n=0}^{\infty} |a_n z^n| < \infty,$$

which implies that $\sum \operatorname{Re} a_n z^n$ converges absolutely whenever $|z| < R$. Therefore, $\sum \operatorname{Re} a_n z^n$ has radius of convergence at least R.

Problem 8 (§3.2, 17). a) We know that a power series with analytic terms is analytic on exactly the disc where it converges absolutely, and that

$$\sum_{n=1}^{\infty} \frac{\sin nz}{2^n} = \sum_{n=1}^{\infty} \frac{e^{inz} - e^{-inz}}{2i \cdot e^{n \log 2}} = \frac{1}{2} \left(\sum_{n=1}^{\infty} e^{inz - n\log 2} - \sum_{n=1}^{\infty} e^{-inz - n\log 2} \right).$$

If we let $z := a + bi$ for $a, b \in \mathbb{R}$, then since $|e^{ir}| = 1$ for every real number r, we can see that

$$\sum_{n=1}^{\infty} |e^{inz - n\log 2}| = \sum_{n=1}^{\infty} |e^{ain - bn - n\log 2}| = \sum_{n=1}^{\infty} (e^{-b-\log 2})^n,$$

which is a geometric series that converges iff $e^{-b-\log 2} < 1 \iff b > -\log 2$. Similarly,

$$\sum_{n=1}^{\infty} |e^{-inz - n\log 2}| = \sum_{n=1}^{\infty} |e^{-ain + bn - n\log 2}| = \sum_{n=1}^{\infty} (e^{b-\log 2})^n,$$

which is a geometric series that converges iff $e^{b-\log 2} < 1 \iff b < \log 2$.
Since there is evidently no region on which $\sum |e^{inz-n\log 2}|$ and $\sum |e^{-inz-n\log 2}|$ both diverge, $\sum_{n=1}^{\infty} e^{inz-n\log 2} - \sum_{n=1}^{\infty} e^{-inz-n\log 2}$ can only converge when $\sum |e^{inz-n\log 2}|$ and $\sum |e^{-inz-n\log 2}|$ both converge, that is, when $|b| < \log 2$. An easy application of the triangle inequality confirms that it converges on this entire region.

b) As before, we can write

$$\sum_{n=1}^{\infty} \frac{\sin nz}{n^2} = \sum_{n=1}^{\infty} \frac{e^{inz} - e^{-inz}}{2in^2} = \sum_{n=1}^{\infty} \frac{e^{ain-bn} - e^{-ain+bn}}{2in^2}.$$

By L'Hopital's rule,

$$\lim_{n\to\infty} \left| \frac{e^{inz}}{2in^2} \right| = \lim_{n\to\infty} \frac{e^{bn}}{n^2} = \lim_{n\to\infty} \frac{b^2 e^{bn}}{2},$$

which equals ∞ if $b > 0$ and 0 if $b \leq 0$. As before, there is no region on which $\sum e^{inz}/(2in^2)$ and $\sum e^{-inz}/(2in^2)$ both diverge, which implies that $\sum (e^{inz} - e^{-inz})/(2in^2)$ can only converge when both $\sum e^{inz}/(2in^2)$ and $\sum e^{-inz}/(2in^2)$ converge. Thus only happens when $b = 0$, and since $\mathbb{R}$ contains no open subset of $\mathbb{C}$, the series is analytic nowhere.