Problem 1. *Let Q be a positive number. (a) Find a bounded harmonic function u on the upper half-plane so that*

(i) its boundary value on $-\infty < x < 0$ is $\frac{Q}{2}$,

(ii) its boundary value on $0 < x < 1$ is Q, and

(iii) its boundary value on $1 < x < \infty$ is 0.

Hint: Use the construction technique for the integrand of a Schwarz-Christoffel transformation.

(b) Find a bounded harmonic function φ on the strip $\{ 0 < y < \pi \}$ such that

(i) its boundary value on $\{ y = \pi \}$ is $\frac{Q}{2}$,

(ii) its boundary value on $\{ x < 0, y = 0 \}$ is Q, and

(iii) its boundary value on $\{ x > 0, y = 0 \}$ is 0.

Hint: Find φ as the function u in (a) composed with a linear fractional transformation and the exponential function.

(c) Show that the curves $\{\varphi(x, y) = \text{constant}\}$ are given by $\tan \frac{y}{2} = c \tanh \frac{x}{2}$ for some constant c. Hint: Use $\sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y$.

(d) Interpret (b) and (c) as solving the following problem of fluid flow. There is a 2-dimensional steady irrotational incompressible fluid flow of constant density in the channel represented by the strip $\{ 0 < y < \pi \}$. The fluid enters through a slit represented by the origin at the rate of Q units per unit time so that the flow exits at each end of the channel (represented by $x = -\infty$ and $x = \infty$) at the rate of $\frac{Q}{2}$ units per unit time. Show that the equation of a streamline is given by $\tan \frac{y}{2} = c \tanh \frac{x}{2}$ for some constant c.

Solution. (a) This is a straight Schwarz-Christoffel application. There should be two terms summing of the form

$$u = Im(a \log(z) + b \log(z - 1))$$

since we have the two discontinuous points at z and $z - 1$. Solving a system of linear equations gets that

$$u(z) = Im(-\frac{Q}{2\pi} \log(z) + \frac{Q}{2\pi} \log(z - 1))$$

is what we want in order for the boundary conditions to be satisfied. We need to pick a branch cut for each of course - and the simple $0 \le \theta \le \pi$ on the upper half plane suffices.

(b) This is immediate from (a) since $z_1 = e^z$ maps the domain given exactly to the condition in (a). Thus, the answer is

$$\varphi(z) = Im(-\frac{Q}{2\pi}\log(e^z) + \frac{Q}{2\pi}\log(e^z - 1))$$

(c) This is just some algebraic manipulation. One can show that the answer we got in (b) to be equivalent to:

$$\varphi(z) = -\frac{Q}{\pi} Im(\log(2\sinh(z/2))).$$

The imaginary part of a logarithm is the angle. Note that we may write $2\sinh(z/2)$ as $2\sinh(x/2)\cos(y/2) + 2i\cosh(x/2)\sin(y/2)$. For the angle θ to be constant, we just need $\tan(\theta)$, or

$$\frac{\cosh(x/2)\sin(y/2)}{\sinh(x/2)\cos(y/2)} = \frac{\tanh(x/2)}{\tan(y/2)}$$

to be constant. But this is exactly what the problem asks for.

(d) This part of the problem is given to make sure you see the connection between the physical conditions of the problem and the mathematical conditions. Notice that if we have the velocity potential $F = \psi + i\varphi$, then $\varphi(a) - \varphi(b)$ equals the flux between them (recall the last problem on the previous problem set).

However, our physical situation exactly reflects this; with the boundary conditions for φ as given, we have exactly the following conditions:

$$\varphi(x < 0) - \varphi(x > 0) = Q \quad (1)$$

$$\varphi(y = \pi) - \varphi(x > 0) = Q/2 \quad (2)$$

$$\varphi(x < 0) - \varphi(y = \pi) = Q/2. \quad (3)$$

Respectively, this means that the fluid is intering at the origin at Q unites per unit time, and it exits each end of the tunnel at $Q/2$ units per time (also, the fact that φ is constant on the two boundaries otherwise means that those are streamlines and the water is trapped inside the channel). Thus, we have exactly the mathematical conditions outlined in part (b), so our calculation in part (c) applies to tell us that the equation of a streamline is given by lines where $\frac{\tanh(x/2)}{\tan(y/2)}$ is constant, as desired.

□

Y.Z.'s notes. Again, a standard implementation of technique. I think understanding (d) is the most important part of this problem - so you have an idea of what intuitively is happening with the physics when you get a weird condition like in (a) or (b). □

Problem 2. *Let* $0 < k < 1$. *Let*

$$K = \int_{t=0}^{1} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},$$

$$iK' = \int_{t=1}^{\frac{1}{k}} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

Let R *be the open rectangle with vertices at*

$$K, \quad K + iK', \quad -K + iK', \quad -K.$$

Let $g(\zeta)$ *be a branch for the function*

$$\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}$$

on the upper half-plane. Consider the Schwarz-Christoffel transformation

$$w = \int_0^z \frac{d\zeta}{g(\zeta)}.$$

(a) Verify that for some choice of the branch $g(\zeta)$ *the Schwarz-Christoffel transformation maps the upper half-plane in the* z *variable one-one onto the rectangle* R *in the* w *variable.*

(b) Describe how that particular branch of $g(\zeta)$ *is defined (i.e., what cuts have to be made in* $\mathbb{C}$ *and what the ranges of the numerical values of the angles in polar representations are).*

(c) Describe the correspondence between the quadruple $\left\{1, -1, \frac{1}{k}, -\frac{1}{k}\right\}$ *of points in the* z *variable and the four vertices of* R *in the* w *variable (i.e., which point in the quadruple goes to which vertex of* R*).*

The definite integrals K *and* K' *are known as complete elliptic integrals of the first kind.*

Solution. (CONSULT OTHER FILE) □

Problem 3. *Let* $h > 0$. *Let* Ω *be the domain in* $\mathbb{C}$ *with variable* $w = u + iv$ *obtained by removing the rectangle* $\{0 < v \leq h,\, u \leq 0\}$ *from the open upper half-plane* $\{v > 0\}$. *Consider the Schwarz-Christoffel transformation from the open upper half-plane in the* z *variable to the domain* Ω *in the* w *variable whose derivative* $\frac{dw}{dz}$ *is given by*

$$\frac{dw}{dz} = A\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}},$$

where A *is a nonzero complex number. Verify that for some nonconstant complex number* A *the Schwarz-Christoffel transformation can be written in the following form*

$$w = \frac{h}{\pi}\left((z+1)^{\frac{1}{2}}(z-1)^{\frac{1}{2}} + \log\left(z + (z+1)^{\frac{1}{2}}(z-1)^{\frac{1}{2}}\right)\right),$$

where

(i) the branch of $(z+1)^{\frac{1}{2}}$ *is chosen with* $0 \leq \arg(z+1) \leq \pi$*,*

(ii) the branch of $(z-1)^{\frac{1}{2}}$ *is chosen with* $0 \leq \arg(z-1) \leq \pi$*, and*

(iii) the branch of log *is the principal branch with the argument defined between* $-\pi$ *and* π*.*

Moreover, verify that that particular Schwarz-Christoffel transformation maps

(i) the interval $(-\infty, -1]$ *in the* z *variable to the line-segment*

$$\{-\infty < u \leq 0,\, v = h\}$$

in the w *variable.*

(ii) the interval $[-1, 1]$ *in the* z *variable to the line-segment*

$$\{\, u = 0,\, 0 \leq v \leq h\,\}$$

in the w *variable, and*

(iii) the interval $[1, \infty)$ *in the* z *variable to the line-segment* $[0, \infty)$ *in the* w *variable.*

Solution. To check that this is what we want, we just need to compute $d\omega/dz$. It turns out to be $(h/\pi)(z+1)^{1/2}(z-1)^{-1/2}$ by direct computation. Also, $\omega(1) = 0$, so we know we can write ω as $A\int_1^z (t+1)^{1/2}(t-1)^{-1/2}dt$ for some A (in fact, $A = h/\pi$.

Since we have the form of a Schwarz-Christoffel transformation, we'll get three straight lines with two turning points (three, counting the point at infinity). Thus, to verify that the transformation maps the intervals to the segments as required by the problem, it really suffices to show that the critical points match up with the points we have chosen to turn at. An explicit calculation gives $\omega(-1) = ih$ and we already know that $\omega(1) = 0$ from earlier. Since we know exactly what the rotations are (at -1 we turn $-\pi/2$ and at 1 we turn $\pi/2$, which we can verify by looking at the branch cuts we chose), this determines that the intervals go exactly where we ant them to be, so we are done. □

Problem 4. *Show that, if* α *and* $\beta \neq 0$ *are real numbers, the equation*

$$z^{2n} + \alpha^2 z^{2n-1} + \beta^2 = 0$$

has $n-1$ *roots with positive real parts if* n *is odd, and* n *roots with positive real parts if* n *is even.*

Hint: Apply the argument principle to the right half-disk of radius R *and let* $R \to \infty$*.*

Solution. Consider the right-half of a disk of radius R centered at the origin. Its boundary is a clockwise contour from $-Ri$ to Ri followed by a vertical contour from Ri to $-Ri$.

Our polynomial is, when n is odd,

$$(iy)^{2n} + a^2 iy^{2n-1} + b^2 = (b^2 - y^{2n}) + iy^{2n-1}.$$

Here, for a given $y = ci$ the imaginary part is positive if $c > 0$ and negative if $c < 0$. The real part is negative if $c > |b^{1/n}| = m$ or if $c < -m$, positive if it is between. Thus, at Ri we start out at (neg, pos), change to (pos, pos) at mi, (pos, neg) at 0, and become (neg,neg) at $-mi$. Also, note that when R is large, the polynomial is dominated by y^2n, which gives (neg,neg) at $-Ri$ and gets us back to (neg, pos) at Ri after n counterclockwise rotations about 0 when we go through the large contour. Therefore, in total we have $n-1$ rotations counterclockwise, since the straight part of the contour was one clockwise rotation.

For n even, our polynomial becomes

$$(iy)^{2n} + a^2 iy^{2n-1} + b^2 = (b^2 + y^{2n}) - iy^{2n-1}.$$

Now, the real part never changes sign, so we have no change, and get $n - 0 = n$ rotations counterclockwise. Thus, we are done.

□

Y.Z.'s notes. The argument principle is a good technique to learn since it generalizes the intermediate value theorem and is much more exact than the intermediate value theorem. I also think you can get Descartes' Rule of Signs from the argument principle, though it is not immediate. □