Bessel Functions and Vibrating Circular Membrane

Method of Separation of Variables. For a linear partial differential equation $Lu = 0$, we can use the method of separation of variables when the linear partial differential operator L can be written as the sum of two linear partial differential operators P and Q so that P depends only on variables $x_1, \cdots, x_k$ and Q depends only on the complementary set of variables $x_{k+1}, \cdots, x_n$. Another condition which needs to be satisfied before the method of separation of variables can be applied is that the domain D must be the product of a domain G in the space of variables $x_1, \cdots, x_k$ and a domain H in the space of variables $x_{k+1}, \cdots, x_n$.

The idea of the method of separation of variables is to consider first unknown functions u of the form $u = vw$ so that v depends on the variables $x_1, \cdots, x_k$ and w depends on the variables $x_{k+1}, \cdots, x_n$. The equation $Lu = 0$ now becomes $wPv + vQw = 0$ which can be rewritten as

$$\frac{Pv}{v} = -\frac{Qw}{w}$$

so that the left-hand side depends only on the variables $x_1, \cdots, x_k$ and the right-hand side depends only on the variables $x_{k+1}, \cdots, x_n$ and as a consequence both sides must be equal to some constant λ. The logic is that if the partial differential equation $Lu = 0$ is satisfied on the domain $D = G \times H$ for some function u of the form

$$u(x_1, \cdots, x_n) = v(x_1, \cdots, x_k)\, w(x_{k+1}, \cdots, x_n),$$

then there exists a constant λ such that

$$\frac{Pv}{w} = -\frac{Qw}{w} = \lambda,$$

which means that

$$\begin{aligned} Pv - \lambda v = 0 &\quad \text{on} \quad G, \\ Qw + \lambda w = 0 &\quad \text{on} \quad H. \end{aligned}$$

We then find all such special product solutions $u = v_j w_j$ (for $j \in J$) and use yet-to-be-determined coefficients c_j to find a solution $u = \sum_{j\in J} c_j v_j w_j$ which satisfies the prescribed boundary conditions.

In practice, we already use some of the prescribed boundary conditions when we seek special product solutions $u = v_j w_j$ so as to limit the index set J to a countable set. Then we use yet-to-be-determined coefficients c_j to find a solution $u = \sum_{j \in J} c_j v_j w_j$ which satisfies the remaining prescribed boundary conditions.

Wave Equation for Vibrating Circular Membrane. To present the details of the method of separation of variables, we choose to work out the example of the wave equation for a vibrating circular membrane. The circular membrane is given by the disk $\{ 0 \leq r \leq c \}$ of radius $c > 0$ in polar coordinates (r, θ). The displacement of the membrane at time t in the direction perpendicular to the disk is given by $u(t, r, \theta)$. The wave equation for the vibrating membrane is given by

$$u_{tt} = a^2 \Delta u$$

with boundary conditions

$$\begin{gathered} u = 0 \quad \text{at} \quad r = c \quad \text{and for all } t \geq 0, \\ u_t = 0 \quad \text{at} \quad t = 0, \\ u = f(r, \theta) \quad \text{at} \quad t = 0, \end{gathered}$$

where $a > 0$ is a constant determined by the surface tension of the membrane and $f(r, \theta)$ is a given function on the disk $\{ 0 \leq r \leq c \}$. The Laplacian Δu of u in polar coordinates is given by

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}.$$

The wave equation $u_{tt} = a^2 \Delta u$ can be written as $Lu = 0$ with

$$L = -\frac{1}{a^2} \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}.$$

The partial differential operator L can be written as $P + Q$ so that the partial differential operator

$$P = -\frac{1}{a^2} \frac{\partial^2}{\partial t^2}$$

depends only on the variable t and the partial differential operator

$$Q = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

depends only on the variables r, θ. The domain D in the space of the variables t, r, θ can be written as the product of the domain $\{ t \geq 0 \}$ in the space of the variable t and the domain $\{ 0 \leq r \leq c \}$ of the space of the variables r, θ. We apply the method of the separation of variables and consider special product functions $u = T(t)v(r, \theta)$. For the special product function $u = T(t)v(r, \theta)$ we impose the boundary conditions

$$u = 0 \quad \text{at} \quad r = c \quad \text{and for all } t \geq 0,$$
$$u_t = 0 \quad \text{at} \quad t = 0,$$

and save the remaining boundary condition

$$u = f(r, \theta) \quad \text{at} \quad t = 0$$

to be used later to determine the coefficients c_j. The method of separation of variables gives us two equations

$$\frac{d^2}{dt^2} T(t) + \lambda a^2 T(t) = 0,$$
$$\frac{\partial^2}{\partial r^2} v(r, \theta) + \frac{1}{r} \frac{\partial}{\partial r} v(r, \theta) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} v(r, \theta) + \lambda v(r, \theta) = 0,$$

where λ is a constant and must be nonnegative, because $-\Delta$ is a nonnegative differential operator in the sense that the integral of the product of $-\Delta g$ and g is nonnegative for g with compact support as one can see by transforming the integral to the integral of the gradient square of f by using integration by parts.

The solution of the differential equation

$$\frac{d^2}{dt^2} T(t) + \lambda a^2 T(t) = 0$$

with the boundary condition $T_t = 0$ for $t = 0$ gives (up to a nonzero constant multiple) the solution

$$T(t) = \cos\left(a\sqrt{\lambda}\, t\right)$$

for each λ.

Though the partial differential operator

$$Q = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

is not the sum of two partial differential operators with each one depending only on one of the two variables r, θ, yet we can change

$$\frac{\partial^2}{\partial r^2} v\left(r,\theta\right) + \frac{1}{r}\frac{\partial}{\partial r} v\left(r,\theta\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} v\left(r,\theta\right) + \lambda v\left(r,\theta\right) = 0$$

to the new partial differential equation

$$r^2 \frac{\partial^2}{\partial r^2} v\left(r,\theta\right) + r\frac{\partial}{\partial r} v\left(r,\theta\right) + \frac{\partial^2}{\partial \theta^2} v\left(r,\theta\right) + \lambda r^2 v\left(r,\theta\right) = 0$$

so that the partial differential operator

$$r^2 \frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2} + \lambda r^2$$

is the sum of the two partial differential operator

$$r^2 \frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r} + \lambda r^2$$

and

$$\frac{\partial^2}{\partial \theta^2}$$

with each one depending on only one of the two variables r, θ. We can now apply again the method of separation of variables and consider special product functions $v\left(r,\theta\right) = R(r)\Theta\left(\theta\right)$. We get two differential equations

$$r^2 \frac{d^2 R}{dr^2} + r\frac{dR}{dr} + \lambda r^2 R - \mu R = 0,$$
$$\frac{d^2 \Theta}{dl\theta^2} + \mu\Theta = 0,$$

where μ is a constant and must be nonnegative, because the operator

$$-\frac{d^2}{d\theta^2}$$

is nonnegative as one can see by integration by parts. Recall that we still have the boundary condition $R(c) = 0$. There are one other boundary condition for $R(r)$ and two boundary conditions for $\Theta(\theta)$, which are not as obvious. The other boundary condition for $R(r)$ is that $R(r)$ is finite at $r = 0$. As we will see later that this is indeed an important boundary condition for $R(r)$, because the differential equation

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}+\lambda r^2R-\mu R=0$$

is not regular at $r=0$ in the sense that the coefficient r^2 for the term $\frac{d^2R}{dr^2}$ of the highest-order differentiation is not nonzero at $r=0$. The two boundary conditions for $\Theta(\theta)$ come from the periodicity of $\Theta(\theta)$ with period 2π. These two boundary conditions are

$$\Theta(0)=\Theta(2\pi),\quad \Theta'(0)=\Theta'(2\pi).$$

For the solution $\Theta(\theta)$ of a second-order differential equation these two conditions are equivalent to the the periodicity of $\Theta(\theta)$ with period 2π.

Second-Order Ordinary Differential Equations with Boundary Conditions Involving Both End-Points. Suppose we have a second-order ordinary differential equation

$$a(x)y''+b(x)y'+c(x)y=0$$

on a finite closed interval $[\alpha,\beta]$ with $a(x)$, $b(x)$, $c(x)$ smooth on $[\alpha,\beta]$ and $a(x)$ nowhere zero on $[\alpha,\beta]$. For the boundary condition $y(\alpha)=0$ and $y'(\alpha)=0$ involving only one single end-point α, there is only one solution $y(x)$ of the differential equation which is identically zero.

However, when we have two homogeneous boundary conditions involving both end-points α and β, for example, the two homogeneous boundary conditions

$$y(\alpha)=0 \text{ and } y(\beta)=0$$

or the two homogeneous boundary conditions

$$y(\alpha)=y(\beta) \text{ and } y'(\beta)=y'(\beta),$$

there may exist another solution $y(x)$ other than the identically zero solution.

The existence of non identically zero solutions for certain homogeneous linear second-order ordinary differential equations subject to two homogeneous boundary conditions involving both end-points makes it possible to apply the method of separation to use complete system of eigenfunctions for differential operators.

In our analysis of the wave equation for a vibrating circular membrane, for a nonnegative number μ we have the second order differential equation

$$\frac{d^2\Theta}{d\theta^2} + \mu\Theta = 0$$

for the unknown function $\Theta(\theta)$ on $[0, 2\pi]$ with the two homogeneous boundary conditions

$$\Theta(0) = \Theta(2\pi), \quad \Theta'(0) = \Theta'(2\pi).$$

involving both end-points 0 and 2π of the interval $[0, 2\pi]$. A general solution of the second-order differential equation is of the form

$$\gamma \sin\left(\sqrt{\mu}\,\theta\right) + \delta \cos\left(\sqrt{\mu}\,\theta\right) \quad \text{with} \quad \gamma, \delta \in \mathbb{R}.$$

The two homogeneous boundary conditions are equivalent to the condition that the solution is periodic of period 2π. In order to have a non identically zero solution, it is necessary and sufficient that $\sqrt{\mu}$ is an integer. Thus the constant μ must be of the form n^2 for some nonnegative integer n. When $\mu = n^2$, the two linearly independent solutions are $\sin n\theta$ and $\cos n\theta$. Each of the two functions $\sin n\theta$ and $\cos n\theta$ is an eigenfunction for the operator

$$-\frac{d^2}{d\theta^2}$$

corresponding to the eigenvalue n^2, because

$$\begin{aligned} -\frac{d^2}{d\theta^2} \sin n\theta &= n^2 \sin n\theta, \\ -\frac{d^2}{d\theta^2} \cos n\theta &= n^2 \cos n\theta, \end{aligned}$$

We know from the theory of Fourier series that the set of all eigenfunctions

$$1, \quad \cos nx, \quad \sin nx \quad \text{for} \quad n \in \mathbb{N}$$

form a complete system of functions on $[0, 2\pi]$ in the sense that any L^2 function (*i.e.,* square-integrable function) on $[0, 2\pi]$ is an infinite linear combination of them in the sense of L^2.

We now use $\mu = n^2$ consider the differential equation

$$r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} + \lambda r^2 R - n^2 R = 0$$

for the unknown function $R(r)$ on $[0, c]$ with the two boundary conditions

$$R(x) \text{ is finite at } x = 0 \quad \text{and} \quad R(c) = 0.$$

We get rid of λ with the rescaling which replaces r by λr as the variable. In other words, we define $y = y(x)$ by $R(r) = y\left(\sqrt{\lambda}\, r\right)$ so that $y(r) = R\left(\frac{1}{\sqrt{\lambda}} r\right)$. Then $y = y(x)$ satisfies the differential equation

$$(\dagger) \qquad x^2 y'' + x y' + \left(x^2 - n^2\right) y = 0,$$

because

$$\begin{aligned} x \frac{dy}{dx}(x) &= \left(\frac{1}{\sqrt{\lambda}} x\right) R'\left(\frac{1}{\sqrt{\lambda}} x\right), \\ x^2 \frac{d^2 y}{dx^2}(x) &= \left(\frac{1}{\sqrt{\lambda}} x\right)^2 R''\left(\frac{1}{\sqrt{\lambda}} x\right) \end{aligned}$$

and

$$\begin{aligned} & x^2 y'' + x y' + \left(x^2 - n^2\right) y \\ = & \left(\frac{1}{\sqrt{\lambda}} x\right)^2 R''\left(\frac{1}{\sqrt{\lambda}} x\right) + \left(\frac{1}{\sqrt{\lambda}} x\right) R'\left(\frac{1}{\sqrt{\lambda}} x\right) \\ & + \left(\lambda \left(\frac{1}{\sqrt{\lambda}} x\right)^2 - n^2\right) R'\left(\frac{1}{\sqrt{\lambda}} x\right). \end{aligned}$$

The boundary conditions become $y(x)$ finite at $x = 0$ and $y(\lambda c) = 0$. The differential equation $(\dagger)$ is known as Bessel's differential equation for the Bessel function of order n. We are going to solve this equation by using a generating function which is a Laurent series in a new indeterminate t whose n-th coefficient is the Bessel function of order n. We are going to write down the generating function and verify that its n-th coefficient satisfies Bessel's differential equation for the Bessel function of order n.

Generating Function for Bessel Functions. We now introduce the generating function for the Bessel functions. It is $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$ and we denotes the n-th coefficient by $J_n(t)$ for $n \in \mathbb{Z}$ so that

$$(*) \qquad e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x) t^n.$$

First we are going to verify that the function $J_n(x)$ satisfies Bessel's differential equation (†). We will do the verification by differentiating $(*)$ with respect to x twice and differentiating $(*)$ with respect to t twice. Differentiating $(*)$ with respect to x once, we get

$$(*)' \qquad \frac{1}{2}\left(t-\frac{1}{t}\right) e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n'(x) t^n.$$

Differentiating $(*)$ with respect to x one more time, we get

$$(*)'' \qquad \left(\frac{1}{2}\left(t-\frac{1}{t}\right)\right)^2 e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n''(x) t^n.$$

Differentiating $(*)$ with respect to t once, we get

$$\frac{x}{2}\left(1+\frac{1}{t^2}\right) e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} n J_n(x) t^{n-1}.$$

and multiplying it by t, we get

$$(*)^\bullet \qquad \frac{x}{2}\left(t+\frac{1}{t}\right) e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} n J_n(x) t^n.$$

Differentiating $(*)^\bullet$ with respect to t one more time, we get

$$\left[\frac{x}{2}\left(1-\frac{1}{t^2}\right) + \frac{x}{2}\left(t+\frac{1}{t}\right)\frac{x}{2}\left(1+\frac{1}{t^2}\right)\right] e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} n^2 J_n''(x) t^{n-1}.$$

and multiplying it by t, we get

$$(*)^{\bullet\bullet} \qquad \left[\frac{x}{2}\left(t-\frac{1}{t}\right) + \left(\frac{x}{2}\left(t+\frac{1}{t}\right)\right)^2\right] e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} n^2 J_n''(x) t^n.$$

We now form the equation

$$x^2(*)'' + x(*)' + x^2(*) - (*)^{\bullet\bullet}$$

whose left-hand side is $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$ times

$$\begin{aligned}
\left(\frac{x}{2}\left(t-\frac{1}{t}\right)\right)^2 + \frac{x}{2}\left(t-\frac{1}{t}\right) - \left[\frac{x}{2}\left(t-\frac{1}{t}\right) + \left(\frac{x}{2}\left(t+\frac{1}{t}\right)\right)^2\right] + x^2 \\
= \left(\frac{x}{2}\left(t-\frac{1}{t}\right)\right)^2 - \left(\frac{x}{2}\left(t+\frac{1}{t}\right)\right)^2 + x^2 \\
= \left(\frac{x}{2}\right)^2\left(t^2-2+\frac{1}{t^2}\right) - \left(\frac{x}{2}\right)^2\left(t^2+2+\frac{1}{t^2}\right) + x^2 = 0
\end{aligned}$$

and whose right-hand side is

$$\sum_{n=-\infty}^{\infty}\left(x^2 J_n''(x) + xJ_n'(x) + \left(x^2-n^2\right)J_n(x)\right)t^n.$$

This is finishes the verification of the differential equation (†) for the Bessel functions $J_n(x)$.

Relation Between Bessel Functions for an Integer and the Negative of the Integer. The generating function $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$ is unchanged when t is replaced by $-\frac{1}{t}$ which from $(*)$ means that

$$J_{-n}(x) = (-1)^n J_n(x) \quad \text{for} \quad n \in \mathbb{Z}.$$

Since Bessel's differential equation (†) is invariant under $n \mapsto -n$, the relation $J_{-n}(x) = (-1)^n J_n(x)$ for $n \in \mathbb{Z}$ precludes the easy way of using $J_{-n}(x)$ to get another solution which is not a scalar multiple of $J_n(x)$. For $n \in \mathbb{Z}$ it will take a more involved procedure to get another solution which is not a scalar multiple of $J_n(x)$. We will indicate how this additional solution is obtained but for the problem of the vibrating circular membrane the additional solution is not needed.

Power Series Expansion for Bessel Functions. We now compute the power series expansion for Bessel functions directly from the power series expansion of the generating function $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}$. We write the generating function as the product two functions so that

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = e^{\frac{xt}{2}}e^{-\frac{x}{2t}}.$$

The power series expansion of the first factor is

$$e^{\frac{xt}{2}} = \sum_{j=0}^{\infty} \frac{x^j}{j!2^j}\,t^j$$

and the power series expansion of the second factor is

$$e^{-\frac{x}{2t}} = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!2^k}\,t^{-k}.$$

We are interested in the coefficient of t_n in the product of these two power series expansion. In order to get t^n as a product we have to consider it as the product of t^j and t^{-k} with $j-k=n$. These the coefficient of t^n in the product of the two power series expansion is the sum of the coefficient of t^j in the first series times the coefficient of t^{-k} in the second series with $j-k=n$. Thus

$$J_n(x) = \sum_{j-k=n} \frac{x^j}{j!2^j}\,t^j\,\frac{(-1)^k x^k}{k!2^k}\,t^{-k}$$

which can be rewritten as

$$(\&) \qquad J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}\left(\frac{x}{2}\right)^{n+2k}$$

when we remove the index j by using $j=n+k$. We would like to remark that at $x=0$ the value of J_0 is 1 but the value of J_n is 0 for any positive integer n.

Bessel Functions of Non-Integral Order and the Other Solution of the Bessel Differential Equation in the Case of Integral Order. When we substitute the power series (&) into the Bessel differential equation, we can readily see that $J_n(x)$ given by (&) satisfies the Bessel differential equation. The same computation shows that if we allow n to be a non-integer and replace $(n+k)!$ by the Gamma function $\Gamma(n+k+1)$, then Bessel's differential equation is satisfied also by $J_n(x)$ given by (&) even when n is not an integer. Recall that the Gamma function is defined by

$$\Gamma(x) = \int_{t=0}^{\infty} e^{-t} t^{x-1} dt.$$

For any real number ν we define the Bessel function by the power series

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{\nu+2k}.$$

Then the Bessel function $J_\nu(x)$ satisfies the Bessel differential equation

$$x^2 y'' + x y' + \left(x^2 - \nu^2\right) y = 0.$$

When ν is not an integer, the two solutions $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent. We would like to mention without going into any further details that for an integer n we can use the following Hankel's function

$$Y_n(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(J_{n+\varepsilon}(x) - (-1)^n J_{-n-\varepsilon}(x)\right)$$

as the second solution of Bessel's differential equation for n. This is the standard technique of taking the limit, as $\varepsilon \to 0$, of a normalized linear combination of two solutions for Bessel's differential equation for $n + \varepsilon$. It turns out that another less obvious linear combination gives a more elegant simpler expression.

$$\frac{1}{2} Y_n(x) + J_n(x)\left(\log 2 - \gamma\right) = J_0(x) \log x + 2 \left(J_2(x) - \frac{1}{2} J_4(x) + \frac{1}{3} J_6(x) - + \cdots \right),$$

where

$$\gamma = \lim_{k \to \infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k} - \log k\right)$$

is Euler's constant. This formula shows that the singularity order of $Y_n(x)$ is $\log x$ as $x \to 0$, which is to be expected, because $Y_n(x)$ is obtained by differentiating

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{\nu+2k}.$$

with respect to ν and

$$\frac{d}{dt}\, t^x = t^x \log x.$$

We conclude that the only solution of Bessel's differential equation (†) for an integer n which has no singularity at $x = 0$ is $J_n(x)$ up to a constant multiple. Thus, to specify that the solution of (†) without singularity has the same effect specifying an initial condition to single out certain solutions of differential equations. The reason why the initial condition takes the form of specifying no singularity at $x = 0$ is because the coefficient of the higher-order term in Bessel's differential equation vanishes at $x = 0$ and is thus a differential equation which is singular at $x = 0$.

Derivatives of Bessel Functions and Recurrent Relation of Bessel Functions. We now discuss the derivatives of Bessel functions and recurrent relation of Bessel functions. In two ways this discussion is needed for the problem of the vibrating circular membrane. The first is that we need some properties of zeroes of $J_n(x)$ for $x \geq 0$. The second is that we need the L^2 norm of $J_n(x)$ with respect to the weight function x over $[0, c]$.

Bessel's differential equation expresses the second-order derivative of the Bessel function in terms of the function and its first-order derivative, but does not give immediately an explicit expression for the first-order derivative of the Bessel function. On the other hand, the generating function, when its first-order derivative with respect to x is compared with its first-order derivative with respect to t, can give formulas for first-order derivatives of the Bessel function as follows. Differentiating the generating function in $(*)$ with respect to x gives $(*)'$ and multiplying both sides by x gives

$$(\flat) \qquad \frac{x}{2}\left(t - \frac{1}{t}\right) e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} x J_n'(x) t^n.$$

Differentiating the generating function in $(*)$ with respect to t and multiplying both sides by x gives $(*)^\bullet$. In order to compare $(\flat)$ with $(*)^\bullet$, we add

$$\frac{x}{t} e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} \frac{x}{t} J_n(x) t^n = \sum_{n=-\infty}^{\infty} x J_{n+1}(x) t^n$$

to $(\flat)$ so that we have

$$(\sharp) \qquad \frac{x}{2}\left(t + \frac{1}{t}\right) e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} \left(x J_n'(x) + x J_{n+1}(x)\right) t^n.$$

We now can equate the right-hand side of $(*)^{\bullet}$ with the right-hand side of $(\sharp)$ and get

$$nJ_n(x) = xJ_n'(x) + xJ_{n+1}(x)$$

or equivalently

$$(\%) \qquad xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x).$$

Replacing n by $-n$, we get

$$xJ_{-n}'(x) = -nJ_{-n}(x) - xJ_{-n+1}(x).$$

using $J_{-n} = (-1)^n J_n(x)$, we get

$$(\$) \qquad xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x).$$

A more compact way of writing these two formulas for the first-order derivatives of Bessel functions is

$$\left(\frac{J_n}{x^n}\right)' = -\frac{J_{n+1}}{x^n},$$
$$(x^n J_n)' = x^n J_{n-1}.$$

Eliminating $J_n'(x)$ from (%) and ($), we get the following algebraic recurrent formula for Bessel functions.

$$xJ_{n+1}(x) = 2nJ_n(x) - xJ_{n-1}(x).$$

Zeros of Bessel Function of Order Zero. Recall that we have the boundary condition $R(c) = 0$. With $R(c) = y\left(\sqrt{\lambda}\,c\right)$ the boundary condition becomes $y\left(\sqrt{\lambda}\,c\right) = 0$. Now $y(x) = J_n(x)$. So we have to formulate the boundary condition in terms of J_n and it becomes $J_n\left(\sqrt{\lambda}\,c\right) = 0$. In order to determine the constant λ, we have to locate the zeroes of J_n for each nonnegative integer n. Let us first consider the simplest case of $J_0(x)$. The Bessel differential equation for it is

$$x^2 y'' + xy' + \left(x^2\right) y = 0$$

which is the same as

$$xy'' + y' + xy = 0.$$

We can get rid of the term involving the first-order derivative by introducing a new dependent variable $u(x) = \sqrt{x}\, y$. Then $u' = \frac{1}{2\sqrt{x}}\, y + \sqrt{x}\, y'$ and

$$u'' = -\frac{1}{4x\sqrt{x}}\, y + \frac{1}{\sqrt{x}}\, y' + \sqrt{x}\, y''$$

so that

$$u'' = -\frac{1}{4x\sqrt{x}}\, y + \frac{1}{\sqrt{x}}\, (y' + xy'') = -\left(\sqrt{x} + \frac{1}{4x\sqrt{x}}\right) y$$

which is the same as

$$u''(x) = -\left(1 + \frac{1}{4x^2}\right) u(x).$$

When x is large, this can be compared with

$$v''(x) = -v(x)$$

which admits $v(x) = \sin x$ as a solution. The key technique is to use the derivative of the Wronskian

$$\begin{vmatrix} u & v \\ u' & v' \end{vmatrix}.$$

In general the Wronskian of n functions $f_1, \cdots, f_n$ is

$$\begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

and is used to investigate the linear independence of $f_1, \cdots, f_n$. The derivative of the Wronskian

$$\begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$$

is

$$(uv' - vu')' = uv'' - vu'',$$

which is equal to

$$uv - v\left(1 + \frac{1}{4x^2}\right)u = -\frac{uv}{4x^2}.$$

Integrating over $[a, b]$ yields

$$\int_a^b \frac{uv}{4x^2}\, dx = vu' - uv'\Big|_{x=a}^{x=b}.$$

We choose $a = 2k\pi$ and $b = (2k+1)\pi$ for $k \in \mathbb{Z}$. Since

$$v(a) = v(b) = 0, \quad v'(a) = 1, \quad v'(b) = -1$$

for our choice of $v(x) = \sin x$, it follows that

$$\int_{2k\pi}^{(2k+1)\pi} u(x)\frac{\sin x}{x^2}\, dx = -4\left(u(2k\pi) + u\left((2k+1)\pi\right)\right).$$

Since

$$\frac{\sin x}{x^2} > 0 \quad \text{for} \;\; 2k\pi < x < (2k+1)\pi,$$

it follows by sign consideration that $u(x)$ has at least one zero in the closed interval $[2k\pi,\, (2k+1)\pi]$ of length π. Since $u(x) = J_0(x)$ is a convergent power series on $[2k\pi,\, (2k+1)\pi]$, there can only be a finite number of zeroes for $u(x)$ in $[2k\pi,\, (2k+1)\pi]$. We are only interested in the zeroes of $J_0(x)$ which are positive numbers. All the zeroes of $J_0(x)$ must be simple, because it satisfies a linear homogeneous second-order differential equation, otherwise the uniqueness statement for the second-order differential equation will force $J_0(x)$ to be identically zero. Thus the zeroes

$$\gamma_{0,1},\, \gamma_{0,2},\, \gamma_{0,3},\, \cdots, \gamma_{0,\ell} \cdots$$

of $J_0(x)$ for $x \geq 0$ can be arranged in a strictly increasing sequence which approaches infinity. Remember that $J_0(0) = 1$ and as a result all the non-negative zeroes of $J_0(x)$ are positive. Let $\alpha_{0,\ell} = \frac{\gamma_{0,\ell}}{c}$ so that $J_0(\alpha_{0,\ell}c) = 0$. Let $\lambda_{0,\ell} = {\alpha_{0,\ell}}^2$. Then $R(r) = J_0(\alpha_{0,\ell}\, r)$ satisfies the differential equation

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + \lambda_{0,\ell}\, r^2 R = 0$$

and the two boundary conditions that $R(r)$ is finite at $r = 0$ and $R(c) = 0$. After we rewrite the differential equation as

$$-\frac{d^2R}{dr^2} - r\frac{dR}{dr} = \lambda_{0,\ell}R$$

and conclude that $R\left(r\right) = J_0\left(\alpha_{0,\ell}\, r\right)$ is an eigenfunction of the differential operator

$$-\frac{d^2}{dr^2} - \frac{1}{r^2}\frac{d}{dr}$$

for the eigenvalue $\lambda_{0,\ell} = {\alpha_{0,\ell}}^2$ with the two boundary conditions that $R(r)$ is finite at $r = 0$ and $R(c) = 0$.

By the Sturm-Liouville theorem which we will introduce later, the family of functions $J_0\left(\alpha_{0,\ell}\, x\right)$ for $\ell \in \mathbb{N}$ is a complete orthogonal family of functions on $[0, c]$ with respect to the weight function x. This completeness property will be used to express our sought-after solution of the vibrating circular membrane as an infinite sum of special product solutions obtained by the method of separation of variables.

Zeroes of Bessel's Function for Higher Integral Order. We now use Rolle's theorem and induction on n to investigate the zeroes of $J_n(x)$. The key is the formula for the first-order derivative of $J_n(x)$, which is

$$(\natural) \qquad \left(\frac{J_n}{x^n}\right)' = -\frac{J_{n+1}}{x^n}.$$

By Rolle's theorem, there is at least one zero of J_n' between two consecutive zeroes of J_n. The above formula ($\natural$) tells us that there is at least zero of J_{n+1} between two consecutive zeroes of J_n. Note that from the power series expansion (&) of $J_n(x)$ centered at $x = 0$ we know that the vanishing order of $J_n(x)$ at $x = 0$ is precisely n. Recall that for the problem of the vibrating circular member only the nonnegative zeroes of $J_n(x)$ are of interest.

Since J_n is the solution of a linear homogeneous second-order differential equation, its derivative cannot vanish at any one of its zeros. This means that J_n' must alternate its sign between two consecutive zeroes of J_n for the following reason. For example, if after the zero of $J_n(x)$ at $x = \sigma$ and before the next zero at $= \tau > \sigma$, the sign of $J_n(x)$ is positive, then from the consideration of the difference quotient the derivative $J_n'(x)$ at $x = \sigma$ must be nonnegative and the derivative $J_n'(x)$ at $x = \tau$ must be nonpositive, and since they are both nonzero, the derivative $J_n'(x)$ at $x = \sigma$ must be positive and the derivative $J_n'(x)$ at $x = \tau$ must be negative. By the above formula ($\natural$) we conclude that J_{n+1} must alternate its sign between two consecutive zeroes of J_n.

Just like the case of $J_0(x)$, we have the following similar situation for $J_n(x)$ for any nonnegative integer n. The zeroes

$$\gamma_{n,1},\ \gamma_{n,2},\ \gamma_{n,3},\ \cdots,\gamma_{n,\ell}\ \cdots$$

of $J_n(x)$ for $x \geq 0$ can be arranged in a strictly increasing sequence which approaches infinity. For $n \geq 1$ the origin $x = 0$ is a zero of $J_n(x)$ of order n and $\gamma_{n,1} = 0$ for $n \geq 1$. Let $\alpha_{n,\ell} = \frac{\gamma_{n,\ell}}{c}$ so that $J_n(\alpha_{0,\ell} c) = 0$. Let $\lambda_{n,\ell} = (\alpha_{n,\ell})^2$. Then $R(r) = J_n(\alpha_{n,\ell} r)$ satisfies the differential equation

$$r^2 \frac{d^2 R}{\partial r^2} + r \frac{dR}{\partial r} + \lambda_{n,\ell}\, r^2 R - n^2 R = 0$$

and the two boundary conditions that $R(r)$ is finite at $r = 0$ and $R(c) = 0$. After we rewrite the differential equation as

$$-\frac{d^2 R}{\partial r^2} - r \frac{dR}{\partial r} - \frac{n^2}{r^2} R = \lambda_{n,\ell} R$$

and conclude that $R(r) = J_n(\alpha_{n,\ell} r)$ is an eigenfunction of the differential operator

$$-\frac{d^2}{dr^2} - \frac{1}{r^2}\frac{d}{dr} - \frac{n^2}{r^2}$$

for the eigenvalue $\lambda_{n,\ell} = (\alpha_{n,\ell})^2$ with the two boundary conditions that $R(r)$ is finite at $r = 0$ and $R(c) = 0$.

Again, by the Sturm-Liouville theorem which we will introduce later, the family of functions $J_n(\alpha_{n,\ell} x)$ for $\ell \in \mathbb{N}$ is a complete orthogonal family of functions on $[0, c]$ with respect to the weight function x and this completeness property will be used to express our sought-after solution of the vibrating circular membrane as an infinite sum of special product solutions obtained by the method of separation of variables.

Orthogonality of the Family $J_n(\alpha_{n,\ell}\, x)$ for $\ell \in \mathbb{N}$. While the completeness of the family $J_n(\alpha_{n,\ell}\, x)$ for $\ell \in \mathbb{N}$ over $[0, c]$ with weight function x depends on the theorem of Sturm-Liouville with some sophisticated arguments, the orthogonality property of the family comes from the orthogonality of two eigenfunctions of a self-adjoint operator for two distinct eigenvalues. We now verify the orthogonality property of the family.

Fix a nonnegative integer n. Take $\ell < m$. Let $y(x) = J_n(\alpha_{n,\ell}\, x)$ and $z(x) = J_n(\alpha_{n,m}\, x)$ on $[0, c]$. These two functions satisfy the following two differential equations

$$x^2 \frac{d^2 y}{\partial x^2} + x \frac{dy}{dx} + \lambda_{n,\ell}\, x^2 y - n^2 y = 0,$$
$$x^2 \frac{d^2 z}{dx^2} + x \frac{dz}{dx} + \lambda_{n,m}\, x^2 z - n^2 z = 0,$$

which we can put in "divergence form"

$$(xy')' - \frac{n^2}{x} y = -\lambda_{n,\ell}\, x\, y.$$
$$(xz')' - \frac{n^2}{x}\, z = -\lambda_{n,m}\, x\, y.$$

Multiplying the first equation by z and the second one by y and taking their difference, we get

$$(xy'(x))'\, z(x) - (xz'(x))'\, y(x) = (\lambda_{n,\ell} - \lambda_{n,m})\, xy(x)z(x).$$

Integrating over $0 \leq x \leq c$ and using integration by parts, we get

$$xy'(x)z(x) - xz'(x)y(x)\Big|_{x=0}^{x=c} = (\lambda_{n,\ell} - \lambda_{n,m}) \int_{x=0}^{x=c} xy(x)z(x)\, dx.$$

From the vanishing of $y(x)$ and $z(x)$ at $x = c$ and their finiteness at $x = 0$ it follows that the left-hand side is zero and

$$\int_{x=0}^{x=c} xy(x)z(x)\, dx = 0,$$

which is the orthogonality of $J_n(\lambda_{n,\ell}\, x)$ and $J_n(\lambda_{n,m}\, x)$ over $[0, c]$ with respect to the weight function x for $1 \leq \ell < m$.

Norm of Eigenfunctions of Bessel's Differential Equation for Integral Order. In order to determine the coefficients in the infinite sum which expresses a given function in terms of the family $J_n(\alpha_{n,\ell}\,x)$ for $\ell \in \mathbb{N}$ over $[0, c]$ with weight function x, we need to have the L^2 norm of each member of the family over $[0, c]$ with weight function x. We now compute this L^2 norm.

Fix a nonnegative integer n and fix a positive integer ℓ. Let $y(x) = J_n(\alpha_{n,\ell}x)$. The function $y(x)$ satisfies the differential equation

$$x^2 y'' + xy' + \left((\alpha_{n,\ell})^2 x^2 - n^2\right) y = 0.$$

After multiplying the equation by $2y'$, we can rewrite it as

$$\frac{d}{dx}(xy')^2 + \left((\alpha_{n,\ell})^2 x^2 - n^2\right)\frac{d}{dx}y^2 = 0.$$

Integrating from $x = 0$ to $x = c$ yields

$$(xy')^2 + \left((\alpha_{n,\ell})^2 x^2 - n^2\right) y^2\Big|_{x=0}^{x=c} = 2(\alpha_{n,\ell})^2 \int_{x=0}^{c} xy^2\,dx.$$

Since $y(c) = 0$ and since both $y(x)$ and $y'(x)$ is finite at $x = 0$ and $y(0) = 0$ when $n > 0$, it follows that

$$\left((\alpha_{n,\ell})^2 x^2 - n^2\right) y^2\Big|_{x=0}^{x=c} = 0$$

and

$$(xy')^2\Big|_{x=0}^{x=c} = (cy'(c))^2.$$

Thus we have

$$(cy'(c))^2 = 2(\alpha_{n,\ell})^2 \int_{x=0}^{c} xy^2\,dx$$

and the formula for the L^2 is given by

$$\int_{x=0}^{c} x\left(J_n(\alpha_{n,\ell}\,x)\right)^2\,dx = \frac{c^2}{2}\left(J_n'(\alpha_{n,\ell}\,c)\right)^2.$$

We can get an alternative form by using the formula (%)

$$xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$$

at $x = \alpha_{n,\ell}\, c$ so that

$$\alpha_{n,\ell}\, c\, J_n'\left(\alpha_{n,\ell}\, c\right) = n J_n\left(\alpha_{n,\ell}\, c\right) - \left(\alpha_{n,\ell}\, c\right) J_{n+1}\left(\alpha_{n,\ell}\, c\right)$$

and

$$J_n'\left(\alpha_{n,\ell}\, c\right) = -\, J_{n+1}\left(\alpha_{n,\ell}\, c\right),$$

because $J_n\left(\alpha_{n,\ell}\, c\right) = 0$. Thus

$$\int_{x=0}^{c} x\left(J_n\left(\alpha_{n,\ell}\, x\right)\right)^2\, dx = \frac{c^2}{2}\left(J_{n+1}\left(\alpha_{n,\ell}\, c\right)\right)^2.$$

Final Answer of Problem of Vibrating Circular Membrane. We now have the special product solutions

$$T(t)R(r)\Theta(\theta) = \cos\left(a\alpha_{n,\ell}\, t\right) J_n\left(\alpha_{n,\ell}\, r\right)\begin{cases} 1 \text{ if } n = 0 \\ \cos n\theta \text{ if } n \in \mathbb{N} \\ \sin n\theta \text{ if } n \in \mathbb{N} \end{cases}$$

for $\ell \in \mathbb{N}$. To get to this point, we have already used up the following five boundary conditions

$$\begin{gathered} T'(0) = 0, \\ \Theta(0) = \Theta(2\pi), \\ \Theta'(0) = \Theta'(2\pi), \\ R(r) \text{ finite at } r = 0, \\ R(c) = 0. \end{gathered}$$

There is the following boundary condition $u(t, r, \theta) = f(r, \theta)$ at $t = 0$, which has not been used. Observe that $\cos\left(a\alpha_{n,\ell}\, t\right) = 1$ at $t = 0$. Now we form the $\mathbb{R}$-linear combination of all the special product functions given above with yet-to-be-determined coefficients $A_{n,\ell}$ (for integers $n \geq 0$ and $\ell \geq 1$) and $B_{n,\ell}$ (for integers $n \geq 1$ and $\ell \geq 1$) so that

$$f(r,\theta) = \sum_{\ell=1}^{\infty} \frac{A_{0,\ell}}{2} J_0\left(\alpha_{0,\ell}\, r\right) + \sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} J_n\left(\alpha_{n,\ell}\, r\right)\left(A_{n,\ell}\cos(n\theta) + B_{n,\ell}\sin(n\theta)\right).$$

Note that here the case of $n = 0$ is singled out, because the Fourier series expansion for $f(\theta)$ is of the form

$$\frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right),$$

where

$$A_n = \frac{1}{\pi} \int_{\theta=0}^{2\pi} f(\theta) \cos(n\theta) \, d\theta \quad \text{for} \quad n \geq 0,$$
$$B_n = \frac{1}{\pi} \int_{\theta=0}^{2\pi} f(\theta) \sin(n\theta) \, d\theta \quad \text{for} \quad n \geq 1,$$

and the case for $n = 0$ has to be treated separately. Here for the vibrating circular membrane, we determine the coefficients $A_{n,\ell}$ (for integers $n \geq 0$ and $\ell \geq 1$) and $B_{n,\ell}$ (for integers $n \geq 1$ and $\ell \geq 1$) by first using the formula for the Fourier series coefficients to get

$$g_n(r) = \frac{1}{\pi} \int_{\theta=0}^{2\pi} f(r, \theta) \cos(n\theta) \, d\theta \quad \text{for} \quad n \geq 0,$$
$$h_n(r) = \frac{1}{\pi} \int_{\theta=0}^{2\pi} f(r, \theta) \sin(n\theta) \, d\theta \quad \text{for} \quad n \geq 1,$$

and then for any fixed n using the expansion in terms of the complete family $J_n(\alpha_{n,\ell} x)$ for $\ell \in \mathbb{N}$ to get $A_{n,\ell}$ from $g_n(r)$ and to get $B_{n,\ell}$ from $h_n(r)$. We use the formula

$$\int_{x=0}^{c} x \left(J_n(\alpha_{n,\ell} x)\right)^2 dx = \frac{c^2}{2} \left(J_{n+1}(\alpha_{n,\ell} c)\right)^2$$

for the L^2 norm of $J_n(\alpha_{n,\ell} x)$ to get

$$A_{n,\ell} = \frac{1}{\frac{c^2}{2} \left(J_{n+1}(\alpha_{n,\ell} c)\right)^2} \int_{r=0}^{c} r \, g_n(r) J_n(\alpha_{n,\ell} r) \, dr,$$
$$B_{n,\ell} = \frac{1}{\frac{c^2}{2} \left(J_{n+1}(\alpha_{n,\ell} c)\right)^2} \int_{r=0}^{c} r \, h_n(r) J_n(\alpha_{n,\ell} r) \, dr$$

to get the following final answer $u(t, r, \theta)$ for the problem of the vibrating circular membrane.

$$u(t, r, \theta) = \sum_{\ell=1}^{\infty} \frac{A_{0,\ell}}{2} \cos(a\alpha_{0,\ell} t) \, J_0(\alpha_{0,\ell} r)$$
$$+ \sum_{n=1}^{\infty} \sum_{\ell=1}^{\infty} \cos(a\alpha_{n,\ell} t) \, J_n(\alpha_{n,\ell} r) \left(A_{n,\ell} \cos(n\theta) + B_{n,\ell} \sin(n\theta)\right).$$