

Brachistochrone (Curve of Shortest Time)

Statement of the Problem. The problem is to find a curve Γ from $(x, y) = (0, h)$ to some point on the vertical line $x = a$ such that when a particle sliding down the curve Γ by gravity would reach the vertical line $x = a$ in the shortest time.

Writing Down the Functional From Conservation of Energy. The sum of the kinetic energy and the potential energy should be constant. We can assume (with an appropriate choice of units) that the acceleration due to gravity is 1 and the mass of the particle is 1. The potential energy is given by the ordinate y and the kinetic energy is given by $\frac{1}{2}v^2$ where v is the speed of the particle. Thus $\frac{1}{2}v^2 + y = h$, because the particle is stationary with $v = 0$ at $y = h$. So $v = \sqrt{2(h - y)}$ and the time for the particle to reach the vertical line $x = a$ is

$$\int_{x=0}^a \frac{ds}{v} = \int_{x=0}^a \frac{\sqrt{1 + y'^2}}{\sqrt{2(h - y)}} dx,$$

where s is the arc-length. To minimize this integral, it is the same as minimizing

$$\int_{x=0}^a \frac{\sqrt{1 + y'^2}}{\sqrt{y}} dx$$

after we change the variable y to $h - y$, which does not affect y'^2 .

Conservation of Energy From the Euler-Lagrange Equation. Let

$$F(x, y, y') = \frac{\sqrt{1 + y'^2}}{\sqrt{y}}.$$

Since $F(x, y, y')$ is independent of x , the analog of the conservation of energy gives the first integral

$$y' \frac{\partial F}{\partial y'} - F = C_1.$$

From

$$\frac{\partial F}{\partial y'} = \frac{1}{\sqrt{y}} \frac{y'}{\sqrt{1 + y'^2}}$$

it follows that

$$C_1 = \frac{1}{\sqrt{y}} \frac{y'^2}{\sqrt{1 + y'^2}} - \frac{\sqrt{1 + y'^2}}{\sqrt{y}} = \frac{1}{\sqrt{y}} \frac{-1}{\sqrt{1 + y'^2}}.$$

This means that

$$y(1 + y'^2) = C_2.$$

Substitution by Cotangent Function to Yield Equation of Cycloid. In order to take advantage of the expression $1 + y'^2$, we make the substitution $y' = \cot \varphi$ so that $1 + y'^2 = \operatorname{cosec}^2 \varphi$ and

$$y = \frac{C_2}{\operatorname{cosec}^2 \varphi} = C_2 \sin^2 \varphi.$$

Taking its differential, we get

$$(*) \quad dy = 2C_2 \sin \varphi \cos \varphi d\varphi.$$

From $y' = \cot \varphi$ we get $dx = \tan \varphi dy$ which can now be written as

$$\begin{aligned} dx &= \tan \varphi (2C_2 \sin \varphi \cos \varphi d\varphi) \\ &= 2C_2 \sin^2 \varphi d\varphi \\ &= C_2 (1 - \cos 2\varphi) d\varphi. \end{aligned}$$

Integrating it, we get

$$x = C_2 \left(\varphi - \frac{1}{2} \sin 2\varphi \right) + C_3.$$

We rewrite (*) as

$$dy = C_2 \sin 2\varphi d\varphi$$

whose integration gives

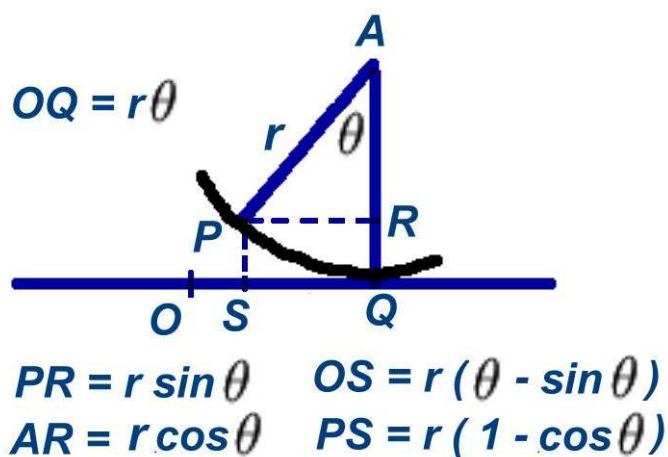
$$y = -C_2 \frac{1}{2} \cos 2\varphi + C_4.$$

Let us choose $\theta = 2\varphi$ so that we do not have to use the factor 2 for variable inside the sine and cosine function. We can choose $\frac{C_2}{2} = r > 0$. We also want the initial point (corresponding to $\theta = 0$) to be the origin. So we can use the following curve

$$x = r(\theta - \sin \theta).$$

We can choose $y = r(1 - \cos \theta)$. These are the parametric equations for the cycloid as seen from the following figure.

It is the locus of a point P on a rolling wheel of radius r . The point P starts at the origin and at the bottom of the wheel. After the wheel turns an angle equal to θ radians, the point of contact between the wheel and the ground is Q . The distance \overline{OQ} between the origin and Q is $r\theta$. From the right-angled triangle $\triangle PRA$ the length \overline{PR} is equal to $r \cos \theta$ and the length \overline{AR} is equal to $r \sin \theta$. The abscissa \overline{OS} of the point P is equal to $r(\theta - \sin \theta)$ and the ordinate \overline{SP} of the point P is equal to $r(1 - \cos \theta)$.



Finally we remember that at the beginning we have replaced y by $h - y$. So we have to turn the cycloid upside down to get the curve of the brachistochrone.