Canonical Transformation

Recall that the canonical equations (or Hamiltonian equations)

$$(*)\qquad \begin{cases} \frac{dy}{dx} = \frac{\partial H}{\partial p} \\ \\ \frac{dp}{dx} = -\frac{\partial H}{\partial y}, \end{cases}$$

where $p = F_{y'}$, and $H = -F + y'p$ come from the reduction of the second-order Euler-Lagrange equation

$$F_y - \frac{d}{dx} F_{y'} = 0$$

to a system of first-order equations and putting it in a form with certain symmetry in y and p via the Legendre transformation. The first equation of $(*)$ is the dual for the definition of $p = F_{y'}$ (which gives the definition of y' in terms of H and p). The second equation of $(*)$ is the rewriting of the Euler-Lagrange equation with $F_{y'}$ replaced by p.

The canonical equations $(*)$ can also be obtained from the usual Euler-Lagrange equation from the extremal problem for the functional

$$(\flat)\qquad \int p\,dy - H dx = \int (p\,y' - H)\,dx$$

with dependent functions p and y of the independent variable t and the function $H\,(y, p, x)$. The usual Euler-Lagrange equation for $(\flat)$ with two dependent functions p and y is

$$\frac{\partial}{\partial p}\,(p\,y' - H(x,y,p)) - \frac{d}{dx}\left(\frac{\partial}{\partial p'}\,(p\,y' - H(x,y,p))\right) = 0,$$
$$\frac{\partial}{\partial y}\,(p\,y' - H(x,y,p)) - \frac{d}{dx}\left(\frac{\partial}{\partial y'}\,(p\,y' - H(x,y,p))\right) = 0,$$

which can be rewritten as

$$y' - \frac{\partial}{\partial p} H(x,y,p) = 0,$$
$$-\frac{\partial}{\partial y} H(x,y,p) - \frac{dp}{dx} = 0.$$

Canonical Transformation and its Generating Function. A canonical transformation is a change of variables from y, p to Y, P so that for some function $H^*(x, Y, P)$ the canonical equations $(*)$ are equivalent to the following system of equations in Y, P of the same form

$$\begin{cases} \frac{dY}{dx} = \frac{\partial H^*}{\partial P} \\ \frac{dP}{dx} = -\frac{\partial H^*}{\partial Y}. \end{cases}$$

One way to construct a canonical transformation is to make the two differentials $p\,dy - H dx$ and $P\,dY - H dx$ differ by an exact differential $d\Phi$ so that

$$p\,dy - H dx = P\,dY - H dx + d\Phi.$$

By collecting the terms involving dx, we can rewrite the equation as

$$(\dagger) \qquad d\Phi = p\,dy - P\,dY + (H^* - H)\,dx$$

and get

$$p = \frac{\partial \Phi}{\partial y}, \quad P = -\frac{\partial \Phi}{\partial Y}, \quad H^* = H + \frac{\partial \Phi}{\partial x}.$$

This gives the transformations between (y, p, H) and (Y, P, H^*). The function $\Phi = \Phi(x, y, Y)$ (which depends only on x, y, Y) is called the generating function of this canonical transformation. A variant form of $(\dagger)$ is

$$(\dagger)' \qquad d\Psi = p\,dy + Y dP + (H^* - H)\,dx,$$

where $\Psi = \Phi + PY$ is the generating function so that the canonical transformation is given by

$$p = \frac{\partial \Psi}{\partial y}, \quad Y = \frac{\partial \Psi}{\partial P}, \quad H^* = H + \frac{\partial \Psi}{\partial x}.$$

The generating function $\Psi = \Psi(x, Y, P)$ in this variant form depends on the variables x, Y, P.

Action and Hamilton-Jacobi Equation. Recall that originally our problem is the extremal problem for the functional

$$J = \int_{x=x_1}^{x_2} F(x, y, y')\,dx$$

and then we change it to the equivalent problem of finding the extremal for the functional

$$\int (py' - H)\, dx,$$

where $H = py' - F$ so that $py' - H = F$. Let $S(x_1, y_1, x_2, y_2)$ be the integral of

$$J = \int_{x=x_1}^{x_2} F(x, y, y')\, dx$$

along the extremal $y = y(x)$ with $y_1 = y(x_1)$ and $y_2 = y(x_2)$. Recall that the general variation formula is

$$\delta J = \int_{x_1}^{x_2} \left(F_y - \frac{d}{dx} F_{y'} \right) (\partial_t y)\, dx + F_{y'} \delta y \Big|_{x=x_1}^{x=x_2} + (F - y' F_{y'})\, \delta x \Big|_{x=x_1}^{x=x_2}.$$

We consider the special case when the variation is for a family of extremals. From the Euler-Lagrange equation for the extremal the integral

$$\int_{x_1}^{x_2} \left(F_y - \frac{d}{dx} F_{y'} \right) (\partial_t y)\, dx$$

on the right-hand side of the general variation formula vanishes and we are left with

$$\delta J = F_{y'} \delta y \Big|_{x=x_1}^{x=x_2} + (F - y' F_{y'})\, \delta x \Big|_{x=x_1}^{x=x_2},$$

which we can rewrite as

$$\delta S = p \delta y \Big|_{x=x_1}^{x=x_2} - H \delta x \Big|_{x=x_1}^{x=x_2},$$

because $p = F_{y'}$ and $H = y' F_{y'} - F$. This means that

$$(\natural) \qquad dS = (p_2 dy_2 - H(x_2, y_2, p_2)\, dx_2) - (p_1 dy_1 - H(x_1, y_1, p_1)\, dx_1).$$

Thus

$$\begin{cases} \frac{\partial S}{\partial x_2} = -H(x_2, y_2, p_2), \\ \frac{\partial S}{\partial y_2} = p_2, \\ \frac{\partial S}{\partial x_1} = H(x_1, y_1, p_1), \\ \frac{\partial S}{\partial y_1} = -p_1. \end{cases}$$

We now replace (x_2, y_2) by (x, y) and replace (x_1, y_1) by (a, b) and regard S as a function of (x, y, a, b). Then

$$(\sharp) \qquad \frac{\partial S}{\partial x} + H\left(x, y, \frac{\partial S}{\partial y}\right) = 0,$$

which is known as the Jacobi-Hamilton equation. We now fix a so that S is a function of (x, y, b) and we can rewrite $(\natural)$ as

$$(\natural)' \qquad dS = pdy - H(x, y, p)\, dx - p_1 db.$$

We can set $Y = b$ and use S in $(\natural)'$ as the generating function for a canonical transformation and get $p_1 = P$ and $H^* = 0$. The vanishing of H^* means that $P =$ constant is a solution of the canonical equations. Thus

$$\frac{\partial S}{\partial b} = \text{constant}$$

is a solution of the canonical equations.

As a matter of fact, even if we do not know where S comes from so that we cannot use the general variation formula, as long as we have the Jacobi-Hamilton equation $(\sharp)$ with S depending on a parameter in a nondegenerate manner whose meaning we will specify later, we can always define

$$p = \frac{\partial S}{\partial y}$$

and set $Y = b$ and define

$$-P = \frac{\partial S}{\partial b}$$

to get

$$dS = pdy - H(x, y, p)\, dx - PdY.$$

When we use S as the generating function for the canonical transformation, we get $H^* = 0$ and the solution of the canonical equation is given by both Y and P being constant. In other words,

$$(\%) \qquad \frac{\partial S}{\partial Y} = P$$

with Y and P constant would give us the equation in x, y, p as the solution for the canonical equation. We now explain precisely what we mean by the dependence of S on b being in a nondegenerate manner. It is to make sure that the equation (%) can be solved for y as a function for x. By the implicit function theorem it suffices to assume that the partial derivative of the left-hand side of (%) with respect to the dependent variable y is nonzero, which is the same as saying that

$$\frac{\partial^2 S}{\partial b \partial y} \neq 0,$$

because Y is defined to be b.