Catenary (Hanging Cable)

Statement of the Problem. The problem is to find the equation of a hanging cable of uniform density whose two end-points are fixed.

Writing Down the Functional From Minimizing the Potential Energy. When the cable achieves equilibrium under the force of gravity, the total potential energy must be minimized. Thus the integral

$$\int_{x=a}^{b} y\,ds = \int_{x=a}^{b} y\sqrt{1+y'^2}\,dx$$

has to be minimized, where s is the arc-length and a, b are the abscissa of the two end-points of the hanging cable.

Conservation of Energy From the Euler-Lagrange Equation. Let

$$F\left(x, y, y'\right) = y\sqrt{1+y'^2}.$$

Since $F\left(x, y, y'\right)$ is independent of x, the analog of the conservation of energy gives the first integral

$$F - y'\frac{\partial F}{\partial y'} = C.$$

From

$$\frac{\partial F}{\partial y'} = y\,\frac{y'}{\sqrt{1+y'^2}}$$

it follows that

$$C = y\sqrt{1+y'^2} - y\,\frac{y'^2}{\sqrt{1+y'^2}} = \frac{y}{\sqrt{1+y'^2}}.$$

This means that

$$y = C\sqrt{1+y'^2}$$

and

$$y' = \sqrt{\frac{y^2-C^2}{C^2}}.$$

Its integration yields

$$x = \int \frac{C\,dy}{\sqrt{y^2-C^2}}.$$

Substitution by Hyperbolic Cosine Function to Yield Equation of Catenary. To evaluate the indefinite integral, we make the substitution $y = C \cosh t$. Then

$$x = \int \frac{C^2 \sinh t \, dt}{\sqrt{C^2 \cosh^2 t - C^2}} = \int C \, dt = Ct + C_1.$$

Finally we end up with the equation

$$y = a \cosh \left(\frac{x - b}{a} \right)$$

for the equation of the hanging cable.

Same Solution for Curve Generating Surface of Revolution of Least Area. We can also interpret the minimization of the integral

$$\int_{x=a}^{b} y \sqrt{1 + y'^2} \, dx$$

as the minimization of the area of the surface of revolution generated by rotating the curve $y = y(x)$ with respect to the x-axis.