Theorem of Cauchy-Goursat and Cauchy's Integral Formula

Differentiable Functions Satisfying Cauchy-Riemann Equation Equivalent to Complex Differentiable. Recall that a real-valued function of two real variables $g(x, y)$ is said to be *differentiable* at a point $(a, b) \in \mathbb{R}^2$ if it can be approximated by a polynomial of degree ≤ 1 at that point to an order higher than the first. More precisely, $g(x, y)$ is differentiable at $(a, b) \in \mathbb{R}^2$ if and only if there exist $\sigma x + \tau y + \rho$ (with $\sigma, \tau, \rho \in \mathbb{R}$) such that

$$g(x, y) = \sigma x + \tau y + \rho + E(x, y)$$

and

$$\lim_{(x,y)\to(a,b)} \frac{E(x, y)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0.$$

A complex-valued function $f(x, y)$ of two real variables x and y is said to be *differentiable* at a point (a, b) if both its real part and its imaginary part (as real-valued functions of two real variables) are differentiable at (a, b). In other words, a complex-valued function $f(x, y)$ of two real variables x and y is differentiable at a point (a, b) there exist complex numbers A and B such that at the point (a, b) the function $f(x, y) - f(a, b)$ can be approximated by $A(x - a) + B(y - b)$ to an order higher than the first.

When a complex-valued function $f(x, y)$ of two real variables x and y is differentiable at a point (a, b) so that, for some complex numbers A and B, at the point (a, b) the function $f(x, y) - f(a, b)$ can be approximated by $A(x - a) + B(y - b)$ to an order higher than the first, by setting $y = b$ (respectively $x = a$), we conclude that $A = f_x(a, b)$ $\big($respectively $B = f_y(a, b)\big)$.

Clearly when $f'(c)$ exists, f is differentiable in this sense with $A = f'(c)$ and $B = i\, f'(c)$. The converse is also true when the Cauchy-Riemann equation is satisfied, namely, if f is differentiable at the point $c = a + ib$ and the Cauchy-Riemann equation is satisfied at c, then $f'(c)$ exists. The differentiability of f at c means that

$$f(x, y) - f(a, b) = f_x(x - a) + f_y(y - b) + E(x, y)$$

with

$$\lim_{z\to c} \frac{E(x, y)}{|z - c|} = 0.$$

The Cauchy-Riemann equation $i\,f_x = f_y$ implies that

$$f(x,y) - f(a,b) = f_x(z-c) + E(x,y)$$

which implies that the limit of the difference quotient

$$\lim_{z\to c} \frac{f(z)-f(c)}{z-c} = f_x.$$

Holomorphic and Harmonic Functions. A function $f(z)$ is said to be *holomorphic* at a point $c \in \mathbb{C}$ if $f'(z)$ exists for every point in some neighborhood of c. Recall the notations

$$\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

. With these notations the Cauchy-Riemann equation $i\,f_x = f_y$ can be rewritten as $\frac{\partial}{\partial \bar z} f = 0$. Moreover, the Laplacian $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ can be written as $4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar z}$ when applied to twice continuously differentiable functions (the twice continuous differentiability being needed to insure the commutativity of partial differentiation). A complex-valued function $h(x,y)$ on some domain D in $\mathbb{C}$ is said to be *harmonic* if its Laplacian $\Delta h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}$ is identically zero on D.

When we represent the Cauchy-Riemann equation by $\frac{\partial}{\partial \bar z} f = 0$ which is the annihilation by a partial differential operator of order one with constant coefficients, we conclude readily that a twice continuously differentiable holomorphic function is harmonic in the sense that its Laplacian is zero.

The Theorem of Cauchy-Goursat. The theorem of Cauchy-Goursat states the following. Let D be a domain in $\mathbf{C}$ and $f(z)$ be a holomorphic function on D. Let C be a simple closed piecewise smooth curve in D such that the domain enclosed by C belongs to D. Then $\int_C f(z)dz$ is zero.

Before we prove the theorem of Cauchy-Goursat, we would like to inject first a remark. Earlier we have seen a special case of this theorem which we called Cauchy's theorem for the smooth case in which we added the additional assumption that the holomorphic function $f(z)$ has continuous first-order partial derivatives up to the boundary of D. The proof of this special case, as explained earlier, is just an immediate application of Stokes's theorem for the following reason. The integral

$$\int_C f(z)dz$$

is a special case of an integral

$$\int_C P(x,y)dx + Q(x,y)dy$$

which is defined as

$$\int_{t=a}^{b} (P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t))dt$$

when C is parametrized by $x = x(t)$ and $y = y(t)$ $(a \leq t \leq b)$. When P and Q are continuously differentiable and $Q_x = P_y$, the integral

$$\int_C P(x,y)dx + Q(x,y)dy$$

vanishes as one can easily see by applying Stokes's theorem to

$$d(P(x,y)dx + Q(x,y)dy) = (Q_x - P_y)dx \wedge dy.$$

Hence from the Cauchy-Riemann equation the theorem of Cauchy-Goursat clearly holds when f is assumed to be continuously differentiable also. What is new in the theorem of Cauchy-Goursat is that the assumption of the continuity of the first-order partial derivatives of f up to the boundary of D is removed so that Stokes's theorem cannot be applied. The idea of the proof of the theorem of Cauchy-Goursat is (i) to divide up the domain D into small pieces and (ii) to approximate on each piece the function f by another function which satisfies the additional assumption so that Cauchy's theorem for the smooth case is applicable to the approximating function, and (iii) finally to show that, as the pieces get smaller and smaller, the difference between the integral for the original function and the sum of the integral over each small piece of the approximating function on that particular small piece goes to zero in the limit.

We now prove the theorem of Cauchy-Goursat without the additional assumption that f is continuously differentiable. By approximating C by a closed broken line whose segments are either horizontal or vertical, we can assume without loss of generality that C is the circumference ∂R_0 of a rectangle R_0 with diameter d and length of circumference ℓ. Suppose

$$\left|\int_{\partial R_0} f(z)dz\right| \geq \varepsilon$$

for some $\varepsilon \geq 0$. By dividing R_0 into 4 equal rectangles, we can pick one R_1 of the 4 rectangles so that

$$\left|\int_{\partial R_1} f(z)dz\right| \geq \frac{\varepsilon}{4}.$$

Again we divide the rectangle R_1 into 4 equal rectangles and so forth. We can find a sequence of nested rectangles $R_k \subset R_{k-1}$ such that

$$\left|\int_{\partial R_k} f(z)dz\right| \geq \frac{\varepsilon}{4^k}.$$

There exists a unique point c belonging to every closed rectangle R_k. Since $f'(c)$ exists, we have

$$f(z) = f(c) + f'(c)(z-c) + E(z)$$

with

$$\lim_{z\to c} \frac{E(z)}{|z-c|} = 0.$$

There exists an open neighborhood U of c such that

$$\frac{E(z)}{|z-c|} < \frac{\varepsilon}{\ell d}$$

on U. There exists k_0 such that for $k \geq k_0$ we have $R_k \subset U$. Since $f(c) + f'(c)(z-c)$ is holomorphic and continuously differentiable, we have

$$\int_{\partial R_k} \left(f(c) + f'(c)(z-c)\right) dz = 0.$$

Hence

$$\left|\int_{\partial R_k} f(z)dz\right| = \left|\int_{\partial R_k} E\, dz\right| < \frac{\varepsilon}{\ell d}\frac{\ell d}{4^k} = \frac{\varepsilon}{4^k},$$

because the diameter of R_k is equal to $\frac{d}{2^k}$ and the length of the circumference of R_k is equal to $\frac{\ell}{2^k}$. Thus we have a contradiction.

Cauchy's Integral Formula. Recall that, by parametrizing the circle $C_{r,c}$ of radius r centered at c by $z = c + re^{i\theta}$ $(0 \leq \theta < 2\pi)$ and by explicitly computing the integral we conclude that

$$\int_{C_{r,c}} (z-c)^n dz$$

is zero when n is not equal to -1 and is equal to $2\pi i$ when n is equal to -1. By the theorem of Cauchy-Goursat or even just the Cauchy theorem for the smooth case, we conclude that the same conclusion holds when $C_{r,c}$ is replaced by a simple closed curve going around c precisely once in the counterclockwise sense.

The *Cauchy integral formula* states the following. Let D be a domain in $\mathbf{C}$ and $f(z)$ be a holomorphic function on D. Let C be a simple closed piecewise smooth curve in D such that the domain enclosed by C belongs to D. Suppose C goes around a point c of D precisely once in the counterclockwise sense. Then

$$f(c) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-c} dz.$$

For the proof of the Cauchy integral formula one considers the function

$$\frac{f(z)-f(c)}{z-c}$$

and circle $C_{r,c}$ of radius r centered at c for some sufficiently small $r > 0$. By the theorem of Cauchy-Goursat

$$\int_C \frac{f(z)-f(c)}{z-c} dz = \int_{C_{r,c}} \frac{f(z)-f(c)}{z-c} dz.$$

We know that the limit of

$$\int_{C_{r,c}} \frac{f(z)-f(c)}{z-c} dz$$

is zero as $r \to 0$ because $\frac{f(z)-f(c)}{z-c}$ is uniformly bounded on a deleted neighborhood of c and the length $2\pi r$ of $C_{r,c}$ goes to zero as $r \to 0$. Hence

$$\int_C \frac{f(z)-f(c)}{z-c} dz$$

which is independent of r must be zero. Thus

$$\int_C \frac{f(z)}{z-c}dz = f(c)\int_C \frac{1}{z-c}dz = 2\pi i\, f(c).$$