Path Integration, Stokes's Theorem, and Cauchy's Theorem for Smooth Functions Satisfying the Cauchy-Riemann Equation

Cauchy-Riemann Equations for Composite Functions. For a complex-valued function $f(z)$ of a complex variable z the Cauchy-Riemann equation is

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y},$$

where $z = x + iy$. An equivalent way to write it is

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f = 0.$$

We introduce the partial differential operator with constant complex coefficients

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

so that the Cauchy-Riemann equation is now written as

$$\frac{\partial}{\partial \bar{z}} f = 0.$$

The reason for the factor $\frac{1}{2}$ in the definition of $\frac{\partial}{\partial \bar{z}}$ is that with this definition its value at the function $f \equiv \bar{z}$ is precisely 1, *i.e.*,

$$\frac{\partial}{\partial \bar{z}}\, \bar{z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(x - iy) \equiv 1.$$

After we rewrite the Cauchy-Riemann equation as getting zero when the partial differential operator $\frac{\partial}{\partial \bar{z}}$ with complex constant coefficients, we conclude readily that rational functions $R(f) = \frac{P(f)}{Q(f)}$ of any function f satisfying the Cauchy-Riemann equation must also satisfy the Cauchy-Riemann equation at points where the denominator $Q(f)$ is nonzero. The reason is that

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{P(f)}{Q(f)}\right) &= \frac{Q(f)\frac{\partial P(f)}{\partial x} - P(f)\frac{\partial Q(f)}{\partial x}}{Q(f)^2} \\ &= \frac{Q(f)P'(f) - P(f)Q'(f)}{Q(f)^2}\,\frac{\partial f}{\partial x}\end{aligned}$$

and

$$\frac{\partial}{\partial y}\left(\frac{P(f)}{Q(f)}\right) = \frac{Q(f)\frac{\partial P(f)}{\partial y} - P(f)\frac{\partial Q(f)}{\partial y}}{Q(f)^2}$$
$$= \frac{Q(f)P'(f) - P(f)Q'(f)}{Q(f)^2}\,\frac{\partial f}{\partial y}$$

from which we get

$$\frac{\partial}{\partial \bar z}\left(\frac{P(f)}{Q(f)}\right) = \frac{Q(f)P'(f) - P(f)Q'(f)}{Q(f)^2}\,\frac{\partial f}{\partial \bar z}$$

which is zero whenever $\frac{\partial f}{\partial \bar z} = 0$.

We can do this in general for composite functions of functions which satisfy the Cauchy-Riemann equation. Consider two functions $g(w)$ and $w = f(z)$ so that both satisfy the Cauchy-Riemann equation. To make the bookkeeping easier to handle, we introduce the the partial differential operator $\frac{\partial}{\partial z}$ with complex constant coefficients which formally is the complex-conjugate of the partial differential operator $\frac{\partial}{\partial \bar z}$ with complex constant coefficients. More precisely,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

so that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar z},$$
$$\frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar z}\right)$$

and the Cauchy-Riemann equation for $f(z)$ is equivalent to

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z}$$

and also equivalent to

$$\frac{\partial f}{\partial y} = i\frac{\partial f}{\partial z}.$$

Moreover, $f'(z) = \frac{\partial f}{\partial z}$ if it exists. Likewise, for $w = u + iv$ we introduce

$$\frac{\partial}{\partial w} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)$$

and

$$\frac{\partial}{\partial \bar{w}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$$

and analogous conclusions hold for the functions of the variable $w = u + iv$ instead of for the functions of the variable $z = x + iy$. For the function $F(z) = g(f(z))$ the chain rule gives

$$\frac{\partial F}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) F$$

$$= \frac{\partial g}{\partial u}\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) u + \frac{\partial g}{\partial v}\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) v$$

$$= \frac{\partial g}{\partial u}\frac{\partial u}{\partial \bar{z}} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial \bar{z}} = \frac{\partial g}{\partial w}\frac{\partial u}{\partial \bar{z}} + i\,\frac{\partial g}{\partial w}\frac{\partial v}{\partial \bar{z}} = \frac{\partial g}{\partial w}\frac{\partial f}{\partial \bar{z}} = 0.$$

Thus we conclude that the composite function $F(z) = g(f(z))$ satisfies the Cauchy-Riemann equation if both $g(w)$ and $w = f(z)$ satisfy the Cauchy-Riemann equation.

Path Integration and Cauchy's Theorem for the Smooth Case. Suppose we have two continuously differentiable complex-valued functions $P(x, y)$ and $Q(x, y)$ (up to the boundary) on a domain D with piecewise continuously differentiable boundary C. The path integral $\int_C P(x, y)dx + Q(x, y)dy$ is defined by

$$\int_C P(x, y)dx + Q(x, y)dy = \int_{t=\alpha}^{\beta} \left(P\left(x(t), y(t)\right)\frac{dx}{dt} + Q\left(x(t), y(t)\right)\frac{dy}{dt}\right) dt$$

when C is parametrized by $t \mapsto (x(t), y(t))$ for $\alpha \le t \le \beta$. Stokes's theorem yields

$$\int_C P(x, y)dx + Q(x, y)dy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

For the special case $P(x, y) = f(z)$ and $Q(x, y) = i\,f(z)$

$$\int_C P(x, y)dx + Q(x, y)dy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

$$= \int_D \left(i\,\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy = 2i \int_D \frac{\partial f}{\partial \bar{z}}\, dxdy$$

which is zero if f satisfies the Cauchy-Riemann equation $\frac{\partial f}{\partial \bar{z}} = 0$. For the special case $P(x, y) = f(z)$ and $Q(x, y) = i\, f(z)$ the path integral $\int_C P(x, y)dx + Q(x, y)dy$ can be rewritten as

$$\int_C P(x, y)dx + Q(x, y)dy = \int_C f(z)dx + i\, f(z)dy = \int_C f(z)dz.$$

We thus have the following conclusion (known as the Cauchy theorem)

$$\int_C f(z)dz = 0$$

for any function f which satisfies the Cauchy-Riemann equation $\frac{\partial f}{\partial \bar{z}} = 0$ and whose two partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are both continuous (up to the boundary) on the domain D with piecewise continuously differentiable boundary C.

Some Special Path Integrals. For the application of Cauchy's theorem to the evaluation of definite integrals, some rather easy special path integrals play an important rôle. For $a \in \mathbb{C}$ and $r > 0$ and $n \in \mathbb{Z}$,

$$\oint_{|z-a|=r} (z-a)^n \, dz = \begin{cases} 0 & \text{for } n \neq -1 \\ 2\pi i & \text{for } n = -1 \end{cases}$$

as one can easily compute with the parametrization $\theta \mapsto z = a + re^{i\theta}$ for $0 \leq \theta \leq 2\pi$ as follows.

$$\oint_{|z-a|=r} (z-a)^n \, dz = \int_{\theta=0}^{2\pi} r^n e^{in\theta} i r e^{i\theta} d\theta = i \int_{\theta=0}^{2\pi} r^{n+1} e^{i(n+1)\theta} d\theta$$

which is equal to

$$\left. \frac{r^{n+1}}{n+1} \, e^{i(n+1)\theta} \right|_{\theta=0}^{\theta=2\pi} = 0$$

when $n \neq -1$ and is equal to

$$i \int_{\theta=0}^{2\pi} d\theta = 2\pi i$$

when $n = -1$.

Geometric Interpretation of the Cauchy-Riemann Equation by Complex-Linearity of the Differential Map. The differential df of the map f is given by the Jacobian matrix $\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$ when the basis $\vec{e}_x$, $\vec{e}_y$ for the domain $\mathbb{R}^2$ and the basis $\vec{e}_u$, $\vec{e}_v$ for the image $\mathbb{R}^2$ are being used. The differential map $df : \mathbb{R}^2 \to \mathbb{R}^2$ is by definition $\mathbb{R}$-linear. With such a notation the map df is simply given by the chain rule. That is, the image of $\vec{e}_x$ is $\frac{\partial u}{\partial x}\vec{e}_u + \frac{\partial v}{\partial x}\vec{e}_v$ according to the chain rule. We can regard the domain $\mathbb{R}^2$ with coordinates x, y as a $\mathbb{C}$-vector space by defining the multiplication of $\vec{e}_x$ by i to be $\vec{e}_y$. Likewise we can regard the target $\mathbb{R}^2$ with coordinates u, v as a $\mathbb{C}$-vector space by defining the multiplication of $\vec{e}_u$ by i to be $\vec{e}_v$. The Cauchy-Riemann equation is equivalent to the statement that df is complex-linear. The complex-linearity of df is the same as saying that i times the image of $\xi\vec{e}_x + \eta\vec{e}_y$ under df is equal to the image of the product of i and $\xi\vec{e}_x + \eta\vec{e}_y$ under df. The former is equal to i times

$$(\xi\, u_x + \eta\, u_y)\vec{e}_u + (\xi\, v_x + \eta\, v_y)\vec{e}_v$$

which is the same as

$$-(\xi\, v_x + \eta\, v_y)\vec{e}_u + (\xi\, u_x + \eta\, u_y)\vec{e}_v.$$

The latter is equal to

$$(-\eta\, u_x + \xi\, u_y)\vec{e}_u + (-\eta\, v_x + \xi\, v_y)\vec{e}_v.$$

For both to be equal for all ξ and η the necessary and sufficient condition is that $u_x = v_y$ and $u_y = -v_x$, which is the Cauchy-Riemann equation.

This geometric interpretation of the Cauchy-Riemann equation by complex-linearity has as its immediate consequence the statement that the composite function $F(z) = g(f(z))$ satisfies the Cauchy-Riemann equation if both $g(w)$ and $w = f(z)$ satisfy the Cauchy-Riemann equation, which we earlier proved in a rather involved way using the chain rule.