Definite Integrals Evaluated by Contour Integration of Branches of Holomorphic Functions.

We are going to discuss the evaluation of definite integrals which requires the application of Cauchy's residue theory to branches of holomorphic functions. In particular, we introduce the following two types of integrals, one involving the fractional powers of the complex variable and another involving the logarithm of the complex variable.

As examples, we evaluate the following two integrals.

$$I_1 = \int_0^\infty \frac{x^{-\alpha}}{x+1}\,dx \quad (0<\alpha<1), \qquad\qquad I_2 = \int_0^\infty \frac{\log x}{\left(x^2+1\right)^2}\,dx.$$

For the evaluation of I_1 we use a branch of the function $z^{-\alpha}$ defined by polar coordinates as follows. For $z = re^{i\theta}$ with $0 \le \theta \le 2\pi$ the value of $z^{-\alpha}$ is $r^{-\alpha}e^{-i\alpha\theta}$.

Choose $0 < r < R$. Let C_R be the curve $\theta \mapsto Re^{i\theta}$ for $0 \le \theta \le 2\pi$ and C_r be the curve $\theta \mapsto re^{i\theta}$ for $0 \le \theta \le 2\pi$.

We now integrate

$$f(z) := \frac{z^{-\alpha}}{z+1}$$

over the contour which starts from the real point r, goes along the real axis from r to R and then follows the curve C_R in the counterclockwise sense to get from the real point R back to the same real point R and then goes along the real axis from R to r and then follows the curve C_r in the clockwise sense to get from the real point r back to the same real point r.

We calculate the residue $\mathrm{Res}_{z=-1} f(z)$, which is given by

$$\lim_{z\to-1}\,(z-(-1))\,f(z) = \lim_{z\to-1}\,(z-(-1))\,\frac{z^{-\alpha}}{z+1} = \lim_{z\to-1} z^{-\alpha} = e^{-i\alpha\pi}.$$

By the residue theorem over the contour described above,

$$\int_r^R \frac{x^{-\alpha}}{x+1}\,dx + \int_{C_R} f(z)dz - \int_{C_r} f(z)dz - \int_r^R \frac{e^{-i\alpha 2\pi}x^{-\alpha}}{x+1}\,dx$$
$$= 2\pi i\,\mathrm{Res}_{z=-1} f(z) = 2\pi i\, e^{-i\alpha\pi}.$$

The reason for the last integral on the left-hand side of the equation is that the value of $z^{-\alpha}$ is $x^{-\alpha}e^{-i\alpha 2\pi}$ at the real point x, because at the real point x the angle θ for its polar representation is 2π. As $R \to \infty$,

$$\left|\int_{C_R} f(z)dz\right| \leq \sup_{z\in C_R} |f(z)| \cdot (\text{length of } C_R) \leq \frac{R^{-\alpha}}{R-1} \cdot 2\pi R$$

approaches 0, because $\alpha > 0$. As $r \to 0$,

$$\left|\int_{C_r} f(z)dz\right| \leq \sup_{z\in C_r} |f(z)| \cdot (\text{length of } C_r) \leq \frac{r^{-\alpha}}{1-r} \cdot 2\pi r$$

approaches 0, because $\alpha < 1$. Hence

$$\left(1 - e^{-i\alpha 2\pi}\right) \int_0^\infty \frac{x^{-\alpha}}{x+1}\, dx = 2\pi i\, e^{-i\alpha\pi}$$

and

$$\int_0^\infty \frac{x^{-\alpha}}{x+1}\, dx = \frac{2\pi i\, e^{-i\alpha\pi}}{1 - e^{-i\alpha 2\pi}} = \frac{2\pi i}{e^{i\alpha\pi} - e^{-i\alpha\pi}} = \frac{\pi}{\sin \pi\alpha}.$$

For the evaluation of I_2 we use a branch of the function $\log z$ defined by polar coordinates as follows. For $z = re^{i\theta}$ with $0 \leq \theta \leq \pi$ the value of $\log z$ is $\log r + i\theta$.

Choose $0 < r < R$. Let Γ_R be the curve $\theta \mapsto Re^{i\theta}$ for $0 \leq \theta \leq \pi$ and Γ_r be the curve $\theta \mapsto re^{i\theta}$ for $0 \leq \theta \leq \pi$.

We now integrate

$$g(z) := \frac{\log z}{\left(z^2+1\right)^2}$$

over the contour which starts from the real point r, goes along the real axis from r to R and then follows the curve Γ_R in the counterclockwise sense to get from the real point R back to the real point $-R$ and goes along the real axis from $-R$ to $-r$ and then follows the curve Γ_r in the clockwise sense to get from the real point $-r$ back to the real point r.

We calculate the residue $\mathrm{Res}_{z=i}g(z)$, which is given by

$$\lim_{z\to i} \frac{d}{dz}\left((z-i)^2 g(z)\right) = \lim_{z\to i} \frac{d}{dz}\left((z-i)^2 \frac{\log z}{\left(z^2+1\right)^2}\right)$$

$$= \left(\frac{d}{dz}\frac{\log z}{(z+i)^2}\right)_{z=i} = \left(\frac{1}{z(z+i)^2} - \frac{2\log z}{(z+i)^3}\right)_{z=i}$$

$$= \frac{1}{i(2i)^2} - \frac{2i\frac{\pi}{2}}{(2i)^3} = \frac{i}{4} + \frac{\pi}{8}.$$

By the residue theorem over the contour described above,

$$\int_r^R \frac{\log x}{\left(x^2+1\right)^2}\,dx + \int_{\Gamma_R} g(z)dz - \int_{\Gamma_r} g(z)dz - \int_{-R}^{-r} \frac{\log(-x)+i\pi}{\left(x^2+1\right)^2}\,dx$$
$$= 2\pi i\,\mathrm{Res}_{z=i}\, g(z) = 2\pi i\left(\frac{i}{4}+\frac{\pi}{8}\right) = -\frac{\pi}{2} + \frac{\pi^2 i}{4}.$$

The reason for the last integral on the left-hand side of the equation is that the value of $\log z$ is $\log(-x)+i\pi$ at the real point x for $x<0$, because at the real point x the angle θ for its polar representation is π while the absolute value of x is $-x$. As $R\to\infty$,

$$\left|\int_{\Gamma_R} g(z)dz\right| \le \sup_{z\in\Gamma_R} |g(z)| \cdot (\text{length of } \Gamma_R) \le \frac{\log R+\pi}{\left(R^2-1\right)^2}\cdot \pi R$$

approaches 0. As $r\to 0$,

$$\left|\int_{\Gamma_r} g(z)dz\right| \le \sup_{z\in\Gamma_r} |g(z)| \cdot (\text{length of } \Gamma_r) \le \frac{\log(-r)+\pi}{\left(1-r^2\right)^2}\cdot \pi r$$

approaches 0. Since

$$-\int_{-R}^{-r} \frac{\log(-x)+i\pi}{\left(x^2+1\right)^2}\,dx = \int_r^R \frac{\log x + i\pi}{\left(x^2+1\right)^2}\,dx$$

by making the substitution $x\to -x$, it follows by letting $R\to\infty$ and $r\to 0$ that

$$\int_0^\infty \frac{\log x}{\left(x^2+1\right)^2}\,dx + \int_0^\infty \frac{\log x + i\pi}{\left(x^2+1\right)^2}\,dx = -\frac{\pi}{2} + \frac{\pi^2 i}{4}.$$

By taking the real part and the imaginary part, we obtain

$$2\int_0^\infty \frac{\log x}{\left(x^2+1\right)^2}\,dx = -\frac{\pi}{2}$$

and

$$\int_0^\infty \frac{\pi}{\left(x^2+1\right)^2}\,dx = \frac{\pi^2}{4}.$$

Thus we get the answer

$$\int_0^\infty \frac{\log x}{(x^2+1)^2}\,dx = -\frac{\pi}{4}.$$

As a bonus we also get

$$\int_0^\infty \frac{1}{(x^2+1)^2}\,dx = \frac{\pi}{4}$$

which we do not need.

Evaluation of Certain Values of the Beta Function. We now look at another example which evaluates certain values of the beta function. The definite integral which we consider is the following.

$$\int_{x=0}^1 \frac{dx}{x^\alpha(1-x)^{1-\alpha}} \quad (0<\alpha<1).$$

For the evaluation of this definite integral by residue theory, we introduce the following function $f(z) = z^{-\alpha}(1-z)^{\alpha-1}$. We have to define an appropriate branch for this function. First we choose a branch for the function $z^{-\alpha}$ and then choose a branch for the function $(1-z)^{1-\alpha}$ and then put the two branches together.

To define a branch for $z^{-\alpha}$, we take away the slit $[0,\infty) \subset \mathbb{R}$ so that we restrict the numerical value for the angle θ in the polar representation of $z = re^{i\theta}$ to $0 < \theta < 2\pi$ and for such a restriction of the value of θ the value of $z^{-\alpha}$ is defined to be $r^{-\alpha}e^{-i\alpha\theta}$.

To define a branch for $(1-z)^{\alpha-1}$, we take away the slit $(-\infty,0] \subset \mathbb{R}$ in $\mathbb{C}$ for the complex variable $1-z$ so that we restrict the numerical value for the angle φ in the polar representation of $1-z=\rho e^{i\varphi}$ to $-\pi<\varphi<\pi$ and for such a restriction of the value of φ the value of $(1-z)^{\alpha-1}$ is defined to be $\rho^{\alpha-1}e^{i(\alpha-1)\varphi}$. Note that taking away the slit $(-\infty,0] \subset \mathbb{R}$ in $\mathbb{C}$ for the complex variable $1-z$ is the same as taking away $[1,\infty) \subset \mathbb{R}$ in $\mathbb{C}$ for the complex variable z, because a translation of adding -1 to the variable $1-z$ moves the slit $(-\infty,0] \subset \mathbb{R}$ in $\mathbb{C}$ for the complex variable $1-z$ to the slit $(-\infty,-1] \subset \mathbb{R}$ in $\mathbb{C}$ for the complex variable $-z$ and the transformation $z \mapsto -z$ moves the slit $(-\infty,-1] \subset \mathbb{R}$ in $\mathbb{C}$ for the complex variable $-z$ to the slit $[1,\infty) \subset \mathbb{R}$ in $\mathbb{C}$ for the complex variable z. Note that the definition of $(1-z)^{\alpha-1}$ uses only the numerical value φ of the angle in the polar representation $1-z=\rho e^{i\varphi}$ and we do not have to choose a numerical value for he angle in the polar representation of z.

When we take the product of the branch $z^{-\alpha}$ and the branch $(1-z)^{\alpha-1}$, the slit $[0,\infty)\subset\mathbb{R}$ in $\mathbb{C}$ for the variable z has to be excluded. However, we can put back the slit $(1,\infty)\subset\mathbb{R}$ in $\mathbb{C}$ for the variable z into the domain of definition of the product of the branch $z^{-\alpha}$ and the branch $(1-z)^{\alpha-1}$ for the following reason. When z is just above the slit $(1,\infty)\subset\mathbb{R}$, the value of θ is 0 and the value of φ is $-\pi$ (corresponding to the numerical value of the angle of $1-z$ being $-\pi$) and as a consequence the value of $f(z)=z^{-\alpha}(1-z)^{\alpha-1}$ is $x^{-\alpha}(x-1)^{\alpha-1}e^{-i(\alpha-1)\pi}$. Now we consider the situation when z is just below the slit $(1,\infty)\subset\mathbb{R}$. When z is just below the slit $(1,\infty)\subset\mathbb{R}$, the value of θ is 2π and the value of φ is π (corresponding to the numerical value of the angle of $1-z$ being π) and as a consequence the value of $f(z)=z^{-\alpha}(1-z)^{\alpha-1}$ is

$$\begin{aligned}&x^{-\alpha}e^{-i2\alpha\pi}(x-1)^{\alpha-1}e^{i(\alpha-1)\pi}\\&=x^{-\alpha}(x-1)^{\alpha-1}e^{-i\alpha\pi-i\pi}\\&=x^{-\alpha}(x-1)^{\alpha-1}e^{-i\alpha\pi+i\pi}\end{aligned}$$

which again is equal to $x^{-\alpha}(x-1)^{\alpha-1}e^{-i(\alpha-1)\pi}$. It means that the function $f(z)=z^{-\alpha}(1-z)^{\alpha-1}$ which is holomorphic on $\mathbb{C}-[0,\infty)$ can be extended to be a continuous function on $\mathbb{C}-[0,1]$.

We now use the following statement which will be proved in Appendix A below: A continuous function on a domain which is holomorphic outside a line-segment in the domain must be holomorphic on the entire domain. From this statement it follows that the function $f(z)=z^{-\alpha}(1-z)^{\alpha-1}$ is holomorphic on $\mathbb{C}-[0,1]$. For later use, we would like to remark that, according to the computation above for the value of $f(x)$ with x just above $(1,\infty)$, we have

$$\begin{aligned}\lim_{\substack{x\to\infty\\x\in\mathbb{R}}} x\,f(x)&=\lim_{\substack{x\to\infty\\x\in\mathbb{R}}} x\left(x^{-\alpha}(x-1)^{\alpha-1}e^{-i(\alpha-1)\pi}\right)\\&=e^{-i(\alpha-1)\pi}=-e^{-i\alpha\pi}.\end{aligned}$$

We now go back to the computation of our definite integral

$$\int_{x=0}^{1}\frac{dx}{x^{\alpha}(1-x)^{1-\alpha}}\quad(0<\alpha<1).$$

Let C_R be the circle of radius R centered at the origin in the counterclockwise sense and let Γ_r be composed of the following four pieces: the right-half of the circle $|z-1|=r$ in the counterclockwise sense, the line-segment joining $1+ri$ to ri, the left-half of the circle $|z|=r$ in the counterclockwise sense, and the line-segment joining $-ri$ to $1-ri$. We apply Cauchy's theorem to the holomorphic function $f(z)=z^{-\alpha}(1-z)^{\alpha-1}$ on the domain enclosed by C_R and Γ_r for $R>0$ sufficiently large and for $r>0$ sufficiently small. Then

$$\int_{C_R} f(z)dz = \int_{\Gamma_r} f(z)dz.$$

We now compute the left-hand side as $R\to\infty$ by using the substitution $z=\frac{1}{w}$. We get

$$\int_{C_R} f(z)dz = -\int_{C_{\frac{1}{R}}} f\left(\frac{1}{w}\right)\left(-\frac{dw}{w^2}\right),$$

where on the right-hand we have a minus sign in front of the integral because of the counterclockwise orientation of $z\in C_R$ corresponds to the clockwise orientation of $w\in C_{\frac{1}{R}}$. Since

$$\lim_{w\to 0}\frac{1}{w}f\left(\frac{1}{w}\right)=\lim_{\substack{x\to\infty\\ x\in\mathbb{R}}} x\,f(x) = -e^{-i\alpha\pi},$$

it follows that

$$\frac{1}{w^2}f\left(\frac{1}{w}\right)$$

has a simple pole at $w=0$ whose residue is $-e^{-i\alpha\pi}$.

In order to compute the limit of

$$\int_{\Gamma_r} f(z)dz$$

as $r\to 0$, we determine the value of $f(x)$ for x just above $(0,1)$ and the value of $f(x)$ for x just below $(0,1)$. To compute the value of $f(x)$ for x just above $(0,1)$, we observe that for x just above $(0,1)$ the value of θ is 0 and the value of φ is 0 (corresponding to the numerical value of the angle of $1-z$ being 0) and as a consequence the value of $f(z)=z^{-\alpha}(1-z)^{\alpha-1}$ is $x^{-\alpha}(x-1)^{\alpha-1}$. Likewise, to compute the value of $f(x)$ for x just below $(0,1)$, we observe that for x just below $(0,1)$ the value of θ is 2π and the value of φ is 0 (corresponding to the numerical value of the angle of $1-z$ being 0) and as a consequence the value of $f(z) = z^{-\alpha}(1-z)^{\alpha-1}$ is $x^{-\alpha}e^{-i\alpha 2\pi}(x-1)^{\alpha-1}$. Thus

$$\lim_{r\to 0}\int_{\Gamma_r} f(z)dz = e^{-i\alpha 2\pi}\int_{x=0}^{1}\frac{dx}{x^{\alpha}(1-x)^{1-\alpha}} - \int_{x=0}^{1}\frac{dx}{x^{\alpha}(1-x)^{1-\alpha}}.$$

Thus we end up with

$$\begin{aligned}\int_{x=0}^{1}\frac{dx}{x^{\alpha}(1-x)^{1-\alpha}} &= \frac{2\pi i\left(-e^{-i\alpha\pi}\right)}{e^{-i\alpha 2\pi}-1}\\ &= \frac{2\pi i}{e^{i\alpha\pi}-e^{-i\alpha\pi}} = \frac{\pi}{\sin\alpha\pi}.\end{aligned}$$

This evaluation is related to the Gamma function, the Beta function and their relation to the sine function, and to the evaluation of the integral I_1 given above, as explained in detail in Appendix B below.

Appendix A: Removal of Singularity for Continuous Functions Holomorphic Outside a Line-Segment

Let D be a domain in $\mathbb{C}$ and $f(z)$ be a continuous function on D. Let L be a line-segment in $\mathbb{C}$.

Theorem. If $f(z)$ is holomorphic on $D - L$, then $f(z)$ is holomorphic on D.

Proof. Without loss of generality we can assume that L is the real line $\mathbb{R}$ and that the closed unit disk centered at the origin $\{|z| \leq 1\}$ is contained in D. It suffices to show that $f(z)$ can is holomorphic at the origin 0. For $0 < r < 1$ let Γ_r be boundary of the domain $\{|z| \leq 1, \operatorname{Im} z \geq r\}$ in the counterclockwise sense and let Ξ_r be boundary of the domain $\{|z| \leq 1, \operatorname{Im} z \leq -r\}$ in the counterclockwise sense. Since $f(z)$ is holomorphic on $D - L$ which contains both $\{|z| \leq 1, \operatorname{Im} z \geq r\}$ and $\{|z| \leq 1, \operatorname{Im} z \leq -r\}$, it follows from Cauchy's integral formula and Cauchy's theorem that

$$f(a) = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{f(z)}{z-a} dz \quad \text{and} \quad 0 = \frac{1}{2\pi i} \int_{\Xi_r} \frac{f(z)}{z-a} dz \quad \text{for } \operatorname{Im} a > r \text{ and } |a| < 1$$

and

$$0 = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{f(z)}{z-a} dz \text{ and } f(a) = \frac{1}{2\pi i} \int_{\Xi_r} \frac{f(z)}{z-a} dz \text{ for } \operatorname{Im} a < -r \text{ and } |a| < 1.$$

Thus

$$f(a) = \frac{1}{2\pi i} \left(\int_{\Gamma_r} \frac{f(z)}{z-a} dz + \int_{\Xi_r} \frac{f(z)}{z-a} dz \right) \quad \text{for } |\operatorname{Im} a| > r \text{ and } |a| < 1$$

Since $f(z)$ is continuous on D, it follows that

$$\lim_{r \to 0} \left(\int_{\Gamma_r} \frac{f(z)}{z-a} dz + \int_{\Xi_r} \frac{f(z)}{z-a} dz \right) = \int_{|z|=1} \frac{f(z)}{z-a} dz$$

for $\operatorname{Im} a \neq 0$ and $|a| < 1$. Hence

$$f(a) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z-a} dz$$

for $\operatorname{Im} a \neq 0$ and $|a| < 1$. Since the right-hand side

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z-a} dz$$

is clearly holomorphic as a function of a for $|a| < 1$. By the continuity of f on D, we conclude that $f(z)$ is holomorphic on $|z| < 1$. Q.E.D.

Appendix B: Gamma and Beta Functions and Their Relation to the Sine Function

Definition of Gamma Function. The Gamma function in a real variable is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

for $x > 0$ to make sure that the integral converges at $t = 0$. When $x > 1$, by integration by parts we get

$$\Gamma(x) = \left[-t^{x-1} e^{-t}\right]_{t=0}^{t=\infty} + (x-1) \int_0^\infty t^{x-2} e^{-t} dt = (x-1)\,\Gamma(x-1).$$

From $\Gamma(1) = \int_0^\infty e^{-t} = 1$ it follows that

$$\Gamma(n) = (n-1)!\,.$$

So the Gamma function is the generalization of the factorial function from integer values to real values. The defining formula

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

actually defined $\Gamma(z)$ for $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$.

Beta Function. A similar analog of the generalization of the binomial coefficient

$$\binom{m+n}{m} = \frac{(m+n)!}{m!\,n!}$$

is the Beta function defined by

$$B(x, y) = \frac{\Gamma(x+y)}{\Gamma(x)\,\Gamma(y)}.$$

We are going to derive the formula for the Beta function as a definite integral whose integrand depends on the variables x and y. This is done by reversing the order of integration of a double integral. For $x > 0$ and $y > 0$ we have

$$\Gamma(x)\Gamma(y) = \left(\int_0^\infty t^{x-1} e^{-t} dt\right)\left(\int_0^\infty u^{y-1} e^{-u} du\right).$$

Using the transformation $u = tv$ and then the transformation $w = t\,(1+v)$, we obtain

$$\begin{aligned}\Gamma(x)\Gamma(y) &= \left(\int_0^\infty t^{x-1}e^{-t}dt\right)\left(\int_0^\infty t^y v^{y-1}e^{-tv}dv\right)\\ &= \left(\int_0^\infty v^{y-1}dv\right)\left(\int_0^\infty t^{x+y-1}e^{-t(1+v)}dv\right)\\ &= \left(\int_0^\infty v^{y-1}dv\right)\left(\int_0^\infty \frac{w^{x+y-1}e^{-w}dw}{(1+v)^{x+y}}\right)\\ &= \Gamma\,(x+y)\int_0^\infty \frac{v^{y-1}dv}{(1+v)^{x+y}},\end{aligned}$$

from which it follows that

$$B\,(x,y) = \int_0^\infty \frac{v^{y-1}dv}{(1+v)^{x+y}}.$$

Finally we use the transformation $v = \frac{\lambda}{1-\lambda}$ to get the alternative formulation

$$B\,(x,y) = \int_0^1 \lambda^{x-1}\,(1-\lambda)^{y-1}\,d\lambda,$$

which is symmetric in x and y.

Relation Between Gamma Function, Beta Function and Sine Function. A very useful case for the Beta function is when $x+y=1$ in the above formula, in which case

$$B(x,1-x) = \Gamma(x)\Gamma(1-x) = \int_0^\infty \frac{v^{x-1}dv}{1+v},$$

which by residue calculus applied to the function

$$\frac{z^{x-1}dz}{1+z}$$

integrated over the contour integral of the boundary of the domain

$$\{\,r<|z|<R\,\} - \{\,\mathrm{Re}\,z\geq 0,\ -r\leq \mathrm{Im}\,z\leq r\,\}\,,$$

yields

$$\frac{\pi}{\sin \pi x}$$

(see the evaluation of I_1 described above). Thus we have the following important formula relating the gamma function to the sine function

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

for $0 < x < 1$.