Definite Integrals Evaluated by Contour Integration Over a Half Circle.

We are going to use Cauchy's residue theory over the boundary of a half disk to evaluate definite integrals of the following form.

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx \quad \text{with } \deg P \leq \deg Q - 2,$$
$$\int_{-\infty}^{\infty} \frac{P(x)\cos x}{Q(x)} dx \quad \text{with } \deg P \leq \deg Q - 1,$$
$$\int_{-\infty}^{\infty} \frac{P(x)\sin x}{Q(x)} dx \quad \text{with } \deg P \leq \deg Q - 1,$$

Here the polynomials $P(x), Q(x)$ have real coefficients and are relatively prime. For the first integral the polynomial $Q(x)$ does not have a real zero. For the second integral the zeroes of the polynomial $Q(x)$ are at most of order one and are contained in the zero-set of $\cos x$. For the third integral the zeroes of the polynomial $Q(x)$ are at most of order one and are contained in the zero-set of $\sin x$. The integrals are computed by using the following residue theorem.

Theorem (Residues). Let D be a bounded domain in $\mathbb{C}$ with piecewise smooth boundary C. Let $f(z)$ be a meromorphic function on D which near the boundary of D is continuous up to the boundary of D. Then

$$\oint_C f(z)dz = 2\pi i \sum_{a \in D} \mathrm{Res}_a f,$$

where $\mathrm{Res}_a f$ is the residue of f at a.

Proof. Let the poles of f in D be $a_1, \cdots, a_k$. Let $D_1, \cdots, D_k$ be disjoint closed disks inside D such that D_j is centered at a_j for $1 \leq j \leq k$. Let C_j be the boundary of D_j in the counterclockwise sense. By the theorem of Cauchy-Goursat

$$\oint_C f(z)dz = \sum_{j=1}^{k} \oint_{C_j} f(z)dz = 2\pi i \sum_{j=1}^{k} \mathrm{Res}_{a_j} f.$$

Q.E.D.

For the evaluation of the above definite integrals when $\cos x$ or $\sin x$ appear and $Q(x)$ has some real zeroes, we need the following notion of a half-residue.

Definition. Let $f(z)$ be a holomorphic function on the punctured disk

$$\left\{ z \in \mathbb{C} \,\middle|\, 0 < |z - a| < R \right\}$$

(where $a \in \mathbb{C}$ and $R > 0$) with a simple pole at a. Let $\alpha > 0$ and $C_{r,\alpha}$ be the half circle

$$\left\{ z = a + re^{i\theta} \,\middle|\, \alpha \leq \theta \leq \alpha\pi \right\}$$

in the counterclockwise sense for $r > 0$. The half-residue of f at a is defined as

$$\frac{1}{2\pi i} \lim_{r \to 0} \int_{C_{r,\alpha}} f(z) dz$$

and is equal to $\frac{1}{2}\mathrm{Res}_a f$ which is independent of the choice of α.

The verification of

$$\lim_{r \to 0} \frac{1}{2\pi i} \int_{C_{r,\alpha}} f(z) dz = \frac{1}{2}\mathrm{Res}_a f$$

follows from writing

$$f(z) = \frac{c_{-1}}{z - a} + g(z)$$

with $g(z)$ holomorphic at z and from using the parametrization $\theta \mapsto a + re^{i\theta}$ ($\alpha \leq \theta \leq \alpha + \pi$) for $C_{r,\alpha}$ to evaluate

$$\int_{C_{r,\alpha}} f(z) dz = c_{-1} \int_{C_{r,\alpha}} \frac{dz}{z - a} + \int_{C_{r,\alpha}} g(z) dz.$$

From

$$\int_{C_{r,\alpha}} \frac{dz}{z - a} = \int_{\theta = \alpha}^{\alpha + \pi} \frac{ire^{i\theta} d\theta}{re^{i\theta}} = \pi i$$

and

$$\lim_{r \to 0} \int_{C_{r,\alpha}} g(z) dz = 0$$

it follows that

$$\frac{1}{2\pi i} \int_{C_{r,\alpha}} f(z) dz = \frac{1}{2} c_{-1} = \frac{1}{2}\mathrm{Res}_a f.$$

Integrals of Rational Functions over the Real Line. For

$$\int_{x=-\infty}^{\infty} \frac{P(x) dx}{Q(x)},$$

the integral of

$$\frac{P(z)}{Q(z)}$$

over the contour of the boundary of the upper half disk of radius R centered at the origin as $R \to \infty$ yields

$$(\ddagger\ddagger) \qquad \int_{x=-\infty}^{\infty} \frac{P(x)dx}{Q(x)} = 2\pi i \sum_{\operatorname{Im} z > 0} \operatorname{Res}_{\zeta} \frac{P(z)}{Q(z)}.$$

The integral $\int_{C_R} \frac{P(z)}{Q(z)} dz$ over the half-circle

$$C_R := \left\{ z \in \mathbb{C} \,\middle|\, z = Re^{i\theta},\, 0 \leq \theta \leq 2\pi \right\}$$

of the meromorphic function $\frac{P(z)}{Q(z)}$ goes to zero as $R \to \infty$, because

$$\sup_{z \in C_R} \left| \frac{P(z)}{Q(z)} \right| = O\left(\frac{1}{R^2} \right)$$

(where $O(u)$ is the Landau symbol meaning that the quotient by u is bounded by a constant as $R \to \infty$) and the length of C_R is $O(R)$ and as a consequence

$$\begin{aligned} \left| \int_{C_R} \frac{P(z)}{Q(z)} dz \right| &\leq \left(\sup_{z \in C_R} \left| \frac{P(z)}{Q(z)} \right| \right) (\text{length of } CR) \\ &= O\left(\frac{1}{R^2} \cdot R \right) = O\left(\frac{1}{R} \right) \to 0 \text{ as } R \to 0. \end{aligned}$$

Moreover, by the residue theorem

$$\int_{C_R} \frac{P(z)}{Q(z)} dz + \int_{x=-R}^{R} \frac{P(x)}{Q(x)} = 2\pi i \sum_{\operatorname{Im} z > 0} \operatorname{Res}_z \frac{P(z)}{Q(z)} = 2\pi i \sum_{\substack{\operatorname{Im} z > 0 \\ |z| < R}} \operatorname{Res}_z \frac{P(z)}{Q(z)}$$

which yields the formula $(\ddagger\ddagger)$ as $R \to \infty$.

Example. We now compute

$$\int_{x=-\infty}^{\infty} \frac{dx}{(x^2 + a^2)^3} \quad (a > 0).$$

We use the meromorphic function

$$f(z) := \frac{1}{\left(z^2 + a^2\right)^3}$$

for which there is only one point in the upper half-plane with nonzero residue. That point is i which is a pole of order 3 and the residue at it is given by

$$\begin{aligned}\frac{1}{2!}\left(\frac{d^2}{dz^2}\frac{(z-ai)^3}{\left(z+a^2\right)^3}\right)_{z=ai} &= \frac{1}{2}\left(\frac{d^2}{dz^2}\frac{1}{(z+ai)^3}\right)_{z=ai} \\ = \frac{1}{2}\left(\frac{(-3)(-4)}{(z+ai)^5}\right)_{z=ai} &= \frac{3}{16a^5 i}.\end{aligned}$$

We thus conclude from the above formula that

$$\int_{x=-\infty}^{\infty} \frac{dx}{\left(x^2+a^2\right)^3} = \frac{3\pi}{8a^5}.$$

Integrals of the Product of a Rational Function and Sine or Cosine Function over the Real Line. For

$$\int_{x=-\infty}^{\infty} \frac{P(x)\cos x dx}{Q(x)}$$

and

$$\int_{x=-\infty}^{\infty} \frac{P(x)\sin x dx}{Q(x)},$$

the integral of

$$\frac{P(z)e^{iz}}{Q(z)}$$

over the contour of the boundary of the upper half disk of radius R centered at the origin as $R \to \infty$ yields

$$\int_{x=-\infty}^{\infty} \frac{P(x)\cos x dx}{Q(x)} = \operatorname{Re}\left(2\pi i \sum_{\operatorname{Im} z>0} \operatorname{Res}_\zeta \frac{P(z)e^{iz}}{Q(z)} + \pi i \sum_{\operatorname{Im} z=0} \operatorname{Res}_\zeta \frac{P(z)e^{iz}}{Q(z)}\right),$$

$$\int_{x=-\infty}^{\infty} \frac{P(x)\sin x dx}{Q(x)} = \operatorname{Im}\left(2\pi i \sum_{\operatorname{Im} z>0} \operatorname{Res}_z \frac{P(z)e^{iz}}{Q(z)} + \pi i \sum_{\operatorname{Im} z=0} \operatorname{Res}_z \frac{P(z)e^{iz}}{Q(z)}\right).$$

For this computation the following two new ingredients have to be incorporated.

(i) Since the degree of $Q(z)$ may only be one m ore than that of $P(z)$, to make sure that

$$\int_{C_R} \frac{P(z)e^{iz}}{Q(z)} dz \to 0 \text{ as } R \to 0$$

we have to do one integration by parts by integrating the factor to get e^{iz} first

$$\int_{C_R} \frac{P(z)e^{iz}}{Q(z)} dz$$

$$= \left. \frac{P(z)e^{iz}}{i\,Q(z)} \right|_{z=-R}^{R} - \int_{C_R} \left(\frac{d}{dz} \frac{P(z)}{Q(z)} \right) e^{iz} dz$$

$$= \left. \frac{P(z)e^{iz}}{i\,Q(z)} \right|_{z=-R}^{R} - \int_{C_R} \frac{\left(P'(z)Q(z) - P(z)Q'(z)\right) e^{iz}}{Q(z)^2} dz$$

and then use

$$\left| \frac{\left(P'(z)Q(z) - P(z)Q'(z)\right)}{Q(z)^2} \right| = O\left(\frac{1}{R^2} \right)$$

(from the degree of $P'(z)Q(z) - P(z)Q'(z)$ no more than the degree of $Q(z)^2$ minus 2) and also use

$$\left| e^{iz} \right| = e^{-\operatorname{Im} z} \leq 1$$

(from $\operatorname{Im} z > 0$ on C_R.

(ii) For zero x_0 of $Q(x)$ on the real line $\mathbb{R}$ we have to modify the contour $\mathbb{R} + C_R$ by replacing $[x_0 - r, x_0 + r]$ by the lower half-circle

$$C_{r,x_0} := \left\{ z \in \mathbb{C} \,\middle|\, z = x_0 + re^{i\theta},\ -\pi \leq \theta \leq \pi \right\}$$

of radius $r > 0$ centered at x_0 in the counterclockwise sense. We label the real roots of $Q(x)$ as $\{x_j\}_j$ and choose the index j such that

$$\left\{ x \in \mathbb{R} \,\middle|\, Q(x) = 0, -R \leq x \leq R \right\} = \{x_j\}_{j \in J_R}$$

for $R > 0$. From the residue theorem

$$\int_{C_R} \frac{P(z)e^{iz}}{Q(z)} dz + \int_{[-R,R] - \bigcup_{1 \leq j \leq J_R} [x_j - r, x_j + r]} \frac{P(x)e^{ix}}{Q(x)} dx + \sum_{j=1}^{J_R} \int_{C_{r,x_j}} \frac{P(z)e^{iz}}{Q(z)} dz$$

is equal to

$$2\pi i \sum_{\substack{|z|<R \\ \operatorname{Im} z \geq 0}} \operatorname{Res}_z \frac{P(z)}{Q(z)}.$$

Finally we get our two formulas

$$\int_{x=-\infty}^{\infty} \frac{P(x)\cos x dx}{Q(x)} = \operatorname{Re}\left(2\pi i \sum_{\operatorname{Im} z>0} \operatorname{Res}_\zeta \frac{P(z)e^{iz}}{Q(z)} + \pi i \sum_{\operatorname{Im} z=0} \operatorname{Res}_\zeta \frac{P(z)e^{iz}}{Q(z)}\right),$$

$$\int_{x=-\infty}^{\infty} \frac{P(x)\sin x dx}{Q(x)} = \operatorname{Im}\left(2\pi i \sum_{\operatorname{Im} z>0} \operatorname{Res}_z \frac{P(z)e^{iz}}{Q(z)} + \pi i \sum_{\operatorname{Im} z=0} \operatorname{Res}_z \frac{P(z)e^{iz}}{Q(z)}\right)$$

by letting $R \to \infty$ and $r \to 0$ and using the half-residue theorem

$$\lim_{r\to 0} \int_{C_{r,x_j}} \frac{P(z)e^{iz}}{Q(z)} dz = \pi i \operatorname{Res}_{x_j} \frac{P(z)e^{iz}}{Q(z)}$$

for every real root x_j of $Q(x)$ and then taking the real and imaginary parts of both sides.

Example. We now compute

$$\int_{x=-\infty}^{\infty} \frac{\sin x dx}{x}$$

in the sense that it is the limit of

$$\int_{x=-R}^{R} \frac{\sin x dx}{x}$$

as $R \to \infty$. The answer is, according to the above formula,

$$\operatorname{Im}\left(\pi i \operatorname{Res}_{z=0} \frac{e^{iz}}{z}\right) = \pi.$$