

Conformal Mappings and Application to Electrostatics

We will apply holomorphic functions (conformal mappings) to the problem of finding electrostatic potentials with prescribed constant boundary values for the two dimensional case. First we discuss the properties of certain special kinds of conformal mappings, in particular, linear fractional transformations, and their mapping properties. Then we use a number of examples to illustrate how conformal mappings are used for applications to problems in electrostatics.

Linear Fractional Transformations. A linear fractional transformation is a map of the form

$$z \mapsto w := \frac{az + b}{cz + d}.$$

Its derivative $\frac{dw}{dz}$ is given by

$$\frac{dw}{dz} = \frac{a(cz + d) - c(az + b)}{(cz + d)^2} = \frac{ac - bd}{(cz + d)^2}.$$

To be able to invert it, the necessary condition is that its derivative is not identically zero, *i.e.*, $ad - bc \neq 0$. This is also both necessary and sufficient for invertibility on the extended complex plane $\mathbb{C} \cup \{\infty\}$ which adds one point ∞ to \mathbb{C} so that near the point ∞ we can use the coordinate system $\zeta = \frac{1}{z}$. Another way to describe the linear fractional transformation

$$z \mapsto w := \frac{az + b}{cz + d}$$

is to introduce two complex variables (z_1, z_2) so that $z = \frac{z_1}{z_2}$ and to introduce two complex variables (w_1, w_2) so that $w = \frac{w_1}{w_2}$ and write

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Since $ad - bc$ is the determinant of the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

when $ad - bc \neq 0$ we can invert the matrix and get

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

and

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Thus

$$z = \frac{dw - b}{-cz + a}.$$

Of course, we get the same result by solving for w in terms of z and from

$$w = \frac{az + b}{cz + d}$$

get

$$czw + dw = az + b$$

and

$$dw - b = -cwz + az = z(-cw + a)$$

and finally

$$z = \frac{dw - b}{-cw + a}.$$

Note that though even without the condition $ad - bc \neq 0$ we can still write down the map

$$z = \frac{dw - b}{-cw + a},$$

but putting

$$w = \frac{az + b}{cz + d}$$

into

$$\frac{dw - b}{-cw + a}$$

gives

$$\frac{d \frac{az+b}{cz+d} - b}{-c \frac{az+b}{cz+d} + a} = \frac{\frac{adz+bd-bcz-bd}{cz+d}}{\frac{-acz-bc+acz+ad}{cz+d}} = \frac{(ad - bc)z}{ad - bc}$$

which gives z only when $ad - bc \neq 0$.

Mapping Behavior of Linear Fractional Transformations. To understand the mapping behavior of the linear fractional transformation

$$z \mapsto w := \frac{az + b}{cz + d},$$

we decompose it as the composite of simpler linear fractional transformations and write

$$w = \frac{az + b}{cz + d} = \frac{a}{c} - \frac{ad - bc}{c} \frac{1}{cz + d}$$

just by dividing $az + b$ by $cz + d$ by long division. This is the same as writing the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

as a product of simpler ones as follows.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \frac{a}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{ad-bc}{c} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix}.$$

The linear fractional transformation is a composite of the following three simpler types of linear fractional transformations:

- (i) A translation $z \mapsto z + a$ for some $a \in \mathbb{C}$.
- (ii) A homothety of multiplication by a nonzero constant $z \mapsto az$ for some $a \in \mathbb{C} - \{0\}$.
- (i) The reciprocal map $z \mapsto \frac{1}{z}$.

The mapping properties of the first two types are clear. Now we look at the third one and verify that the reciprocal map maps the collection of circles and straight lines to the collection of circles and straight lines. However, it does not mean that it maps circles to circles and maps straight lines to straight lines. It may happen that the reciprocal map maps some circle to a straight line and maps some straight line to a circle. The equation of the circle centered at $a \in \mathbb{C}$ with radius $r > 0$ is given by

$$|z - a|^2 = r^2.$$

The transformation

$$z \mapsto w := \frac{1}{z}$$

would map it to

$$\left| \frac{1}{w} - a \right|^2 = r^2,$$

which is

$$|1 - aw|^2 = r^2 |w|^2$$

or

$$1 - aw - \bar{a}\bar{w} + |a|^2 |w|^2 = r^2 |w|^2$$

or

$$(|a|^2 - r^2) |w|^2 - aw - \bar{a}\bar{w} + 1 = 0.$$

This is a straight line if and only if $|a|^2 - r^2 = 0$, *i.e.*, the origin 0 is on the original circle

$$\left| \frac{1}{w} - a \right|^2 = r^2.$$

Otherwise, the the original circle is mapped to another circle

$$(|a|^2 - r^2) |w|^2 - aw - \bar{a}\bar{w} + 1 = 0.$$

Let us see what a straight line is mapped to by this linear fractional transformation. The equation of a general line is

$$\alpha x + \beta y + \gamma = 0 \quad \text{with } \alpha, \beta, \gamma \in \mathbb{R}.$$

We can rewrite it in terms of the complex variable z and get

$$\alpha \frac{z + \bar{z}}{2} + \beta \frac{z - \bar{z}}{2i} + \gamma = 0$$

or

$$\left(\frac{\alpha}{2} - \frac{\beta}{2}i \right) z + \left(\frac{\alpha}{2} + \frac{\beta}{2}i \right) \bar{z} + \gamma = 0.$$

Let $a = \frac{\alpha}{2} - \frac{\beta}{2}i \in \mathbb{C}$. Then the equation of the line becomes

$$az + \bar{a}\bar{z} + \gamma = 0,$$

which is transformed by the reciprocal map to

$$\frac{a}{w} + \frac{\bar{a}}{\bar{w}} + \gamma = 0$$

or

$$a\bar{w} + \bar{w} + \gamma|w|^2 = 0.$$

This is a circle if and only if $\gamma \neq 0$, *i.e.*, if and only if the origin is not on the original line

$$\alpha x + \beta y + \gamma = 0,$$

otherwise the image is a circle. The reciprocal map is the composite of the conjugation map $z \mapsto \bar{z}$ (which is the reflection with respect to the real axis) and the *inversion with respect to the unit circle* studied in Euclidean plane geometry

$$z \mapsto \frac{1}{\bar{z}}$$

which sends $z = re^{i\theta}$ to $\frac{1}{r}e^{-i\theta}$ and can be described as sending a point P to another point Q on the line joining the origin to P so that the product of the distance between the origin and P and the distance between the origin and Q is 1.

The rule to remember when we have to determine whether the image of a circle or a straight line should be a circle or a straight line is that a straight line can be regarded as a circle which passes through ∞ .

Cross-Ratios. A general linear fractional transformation

$$z \mapsto w := \frac{az + b}{cz + d}$$

is defined by four complex numbers a, b, c, d . However, the simultaneous multiplication of the four complex numbers by the same nonzero complex number would not change the linear fractional transformation. So the degree of freedom in the choice of a general linear fractional transformation is three complex numbers. When we fix three distinct complex numbers z_1, z_2, z_3 , we expect to be able to find a linear fractional transformation for any prescribed triple of distinct complex numbers w_1, w_2, w_3 . However, when we want to find a linear fractional transformation mapping a prescribed quadruple of distinct complex numbers z_1, z_2, z_3, z_4 to another prescribed quadruple of distinct complex numbers w_1, w_2, w_3, w_4 , we expect not to be able to do it unless the two quadruples satisfy some relation. The relation turns out to be the *preservation of cross-ratios* by any linear fractional transformation. For

any given quadruple of distinct complex numbers z_1, z_2, z_3, z_4 , its *cross-ratio* (z_1, z_2, z_3, z_4) is defined by

$$(z_1, z_2, z_3, z_4) = \frac{\frac{z_1 - z_3}{z_1 - z_4}}{\frac{z_2 - z_3}{z_2 - z_4}}.$$

The verification of the preservation of a cross-ratio by a linear fractional transformation is as follows. We have

$$w_j = \frac{az_j + b}{cz_j + d} \quad \text{for } 1 \leq j \leq 4.$$

Then

$$\begin{aligned} w_1 - w_3 &= \frac{az_1 + b}{cz_1 + d} - \frac{az_3 + b}{cz_3 + d} \\ &= \frac{(az_1 + b)(cz_3 + d) - (az_3 + b)(cz_1 + d)}{(cz_1 + d)(cz_3 + d)} \\ &= \frac{(adz_1 + bcz_3) - (adz_3 + bcz_1)}{(cz_1 + d)(cz_3 + d)} \\ &= \frac{(ad - bc)(z_1 - z_3)}{(cz_1 + d)(cz_3 + d)}. \end{aligned}$$

Changing z_3 to z_4 and w_3 to w_4 and taking the quotient of the two equations, we get

$$\frac{w_1 - w_3}{w_1 - w_4} = \frac{z_1 - z_3}{z_1 - z_4} \frac{cz_4 + d}{cz_3 + d}.$$

Changing z_1 to z_2 and w_1 to w_2 and taking the quotient of the two equations, we get

$$\frac{\frac{w_1 - w_3}{w_1 - w_4}}{\frac{w_2 - w_3}{w_2 - w_4}} = \frac{\frac{z_1 - z_3}{z_1 - z_4}}{\frac{z_2 - z_3}{z_2 - z_4}}.$$

This, in particular, tells us that the linear fractional transformation $z \mapsto w$ which sends a prescribed triple of distinct complex numbers z_1, z_2, z_3 to a prescribed triple of distinct complex numbers w_1, w_2, w_3 must preserve the cross-ratios (z_1, z_2, z_3, z) and (w_1, w_2, w_3, w) , which means that the linear fractional transformation is given by

$$\frac{\frac{w_1 - w_3}{w_1 - w}}{\frac{w_2 - w_3}{w_2 - w}} = \frac{\frac{z_1 - z_3}{z_1 - z}}{\frac{z_2 - z_3}{z_2 - z}}.$$

This is an easy way to write down explicitly the linear fractional transformation.

Conformality of Holomorphic Map With Nonzero Derivative. We now consider the mapping behavior of a holomorphic map $w = f(z)$ at a point z_0 where its derivative $f'(z_0)$ is nonzero. Take a smooth curve $t \mapsto z(t)$ passing through the point z_0 so that $z(0) = z_0$. The image of the curve $t \mapsto w(t) := f(z(t))$ passes through the point $w_0 := f(z_0)$. By the chain rule and the Cauchy-Riemann equation We have

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \\ &= \frac{\partial f}{\partial x} \frac{dx}{dt} + i \frac{\partial f}{\partial x} \frac{dy}{dt} \\ &= \frac{\partial f}{\partial x} \left(\frac{dx}{dt} + i \frac{dy}{dt} \right) \\ &= \frac{\partial f}{\partial x} \frac{dz}{dt} = f'(z) \frac{dz}{dt}. \end{aligned}$$

This means that the angle made between the tangent to the curve $t \mapsto z(t)$ at z_0 and the real axis in the z variable is equal to the angle made between the tangent to the curve $t \mapsto w(t)$ at w_0 and the real axis in the w variable plus the angle of the polar representation of $f'(z_0)$. In particular, the angle made between two curves at z_0 is the same as the angle at w_0 made between their images under the map f . When we have a family of curves $t \mapsto \gamma_s(t)$ parametrized by a real variable s passing through z_0 such that their tangents at z_0 rotates in the counterclockwise sense as s increases, the tangent at w_0 to the image curve $t \mapsto f(\gamma_s(t))$ under f also rotates in the counterclockwise sense. In other words, the holomorphic map $z \mapsto w := f(z)$ is angle-preserving and orientation-preserving when $f'(z_0) \neq 0$. We call a mapping with the angle-preserving property a *conformal* mapping. Note that the conjugation map $z \mapsto \bar{z}$ is conformal but reverses the orientation.

Example One. Consider a “lightning fence” of height h , represented by the line-segment $[0, hi]$ joining 0 and hi . Let $\text{Im } z = 0$ represent the ground. Let D be the domain which is equal to the upper half-plane $\{\text{Im } z > 0\}$ minus the line-segment $[0, hi]$. Assume that there is an electric potential u on the domain D whose value at $\text{Im } z = 0$ and line-segment $[0, hi]$ is 0 and whose value has limit ∞ as $\text{Im } z \rightarrow \infty$. Find u and the electric field $-\text{grad } u$.

Solution. Consider the map $z \mapsto z_1 := z^2$ which maps the domain D to \mathbb{C} minus $\{\text{Im } z = 0, \text{Re } z \geq -h^2\}$. Let $z_2 = z_1 + h^2$ so that $z \mapsto z_2$ maps D

to \mathbb{C} minus $\{\text{Im } z = 0, \text{Re } z \geq 0\}$. Let w be the branch of $\sqrt{z_2}$ defined by the numerical value of the angle in the polar representation of z_2 in $(0, 2\pi)$. We can now set u to be the real part of w which is equal to $\sqrt{z^2 + h^2}$. The electric field $-\text{grad } u$ is given by

$$\begin{aligned} -\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) &= -\left(\frac{\partial u}{\partial x}, -\frac{\partial v}{\partial x}\right) \\ &= -\frac{\partial(u - iv)}{\partial x} = -\frac{dw}{dz} = -\frac{z}{\sqrt{z^2 + h^2}}. \end{aligned}$$

Example Two. Let D be the domain in \mathbb{C} which is obtained from \mathbb{C} by removing the two rays $[1, \infty)$ and $(-\infty, -1]$. Find a bounded harmonic function u on the domain D such that the value of u is 1 on $[1, \infty)$ and is -1 on $(-\infty, -1]$.

Solution. First of all, consider the map

$$z \mapsto z_1 = \frac{z + 1}{z - 1}$$

so that the domain D becomes \mathbb{C} minus $[0, \infty)$ with $(-\infty, -1]$ going to $[0, 1]$ and $[1, \infty)$ going to itself. Apply $z_1 \mapsto z_2 = \sqrt{z_1}$ which is the branch of $\sqrt{z_1}$ defined by the numerical value of the angle in the polar representation of z_1 in $(0, 2\pi)$. In the map $z_1 \mapsto z_2$ the domain \mathbb{C} minus $[0, \infty)$ becomes the upper half-plane and the interval $[1, \infty)$ goes to the union of $[1, \infty)$ and $(-\infty, -1]$ and the interval $[0, 1]$ goes to $[-1, 1]$. We now apply the map $z_2 \mapsto z_3 = \frac{z_2 + 1}{z_2 - 1}$ which maps the upper half-plane to the lower half-plane, because $ad - bc = -2 < 0$ and the interval $[-1, 1]$ goes to $(-\infty, 0]$ and the union of $[1, \infty)$ and $(-\infty, -1]$ goes to $[0, \infty)$. We now take $z_3 \mapsto w = \log z_3$ which is the branch of $\log z_3$ defined by the numerical value of the angle in the polar representation of z_3 in $(-\pi, 0)$. Finally we set $u = 1 + \frac{2}{\pi} \text{Im } w$ which is obtained by solving for α and β so that $\alpha \text{Im } w + \beta$ becomes -1 when $\text{Im } w = -\pi$ and becomes 1 when $\text{Im } w = 0$.

Example Three. Find a harmonic function u on the unit upper half-disk which assumes the value 1 on the upper half-circle and the values 0 on the interval $[-1, 1]$.

Solution. First use a linear fractional transformation

$$w = \frac{z + 1}{z - 1}$$

sending the point 1 to infinity and the point -1 to the origin so that the upper half-disk becomes the third quadrant. Thus the interval $[-1, 1]$ is mapped to the negative real axis and the upper half-circle is mapped to the negative imaginary axis. We can set $u = -\frac{2}{3\pi}(-\pi + \text{Im} \log w)$, where the branch of the log is defined by the condition that the numerical value of the angle in the polar representation of the variable is in $(-\pi, \pi]$.

Bridge Between Exponential Map and Sine and Cosine. The map

$$w = \frac{1}{2} \left(z + \frac{1}{z} \right)$$

serves as the bridge between the exponential map and the sine and cosine, hyperbolic sine, and hyperbolic cosine functions. To understand the mapping behavior of the map, we use polar representations for z . Write $z = re^{i\theta}$. Then we have

$$w = \frac{1}{2} \left(re^{i\theta} + \frac{1}{r}e^{-i\theta} \right) = \frac{1}{2} \left(r + \frac{1}{r} \right) \cos \theta + \frac{1}{2} \left(r - \frac{1}{r} \right) \sin \theta.$$

When we use the Cartesian representation $w = u + iv$ for w , we get

$$\begin{cases} u = \frac{1}{2} \left(r + \frac{1}{r} \right) \cos \theta \\ v = \frac{1}{2} \left(r - \frac{1}{r} \right) \sin \theta. \end{cases}$$

The circle $|z| = r$ for $0 < r < 1$ is mapped to the ellipse

$$\frac{u^2}{\alpha^2} + \frac{v^2}{\beta^2} = 1$$

with semi-major axis $\alpha = \frac{1}{2} \left(r + \frac{1}{r} \right)$ and semi-minor axis $\beta = \frac{1}{2} \left(r - \frac{1}{r} \right)$. When the complex variable z travels along the circle $|z| = r$ in the counter-clockwise sense as θ goes from 0 to 2π , the complex variable w travels along the ellipse

$$\frac{u^2}{\alpha^2} + \frac{v^2}{\beta^2} = 1$$

clockwise sense as θ goes from 0 to 2π .

Now consider the circle $|z| = r$ with $r > 1$. The circle $|z| = r$ for $r > 0$ is mapped to the ellipse

$$\frac{u^2}{\alpha^2} + \frac{v^2}{\gamma^2} = 1$$

with semi-major axis $\alpha = \frac{1}{2} \left(r + \frac{1}{r} \right)$ and semi-minor axis $\gamma = \frac{1}{2} \left(r - \frac{1}{r} \right) = -\beta$. When the complex variable z travels along the circle $|z| = r$ in the counterclockwise sense as θ goes from 0 to 2π , the complex variable w travels along the ellipse

$$\frac{u^2}{\alpha^2} + \frac{v^2}{\gamma^2} = 1$$

also in the counterclockwise sense as θ goes from 0 to 2π .

The unit circle $|z| = 1$ is mapped to the closed interval $[-1, 1]$ twice when the ellipse

$$\frac{u^2}{\alpha^2} + \frac{v^2}{\beta^2} = 1$$

degenerates to $[-1, 1]$ as $\beta \rightarrow 1$ from < 1 .

From the above discussion we know that the map

$$w = \frac{1}{2} \left(z + \frac{1}{z} \right)$$

- (i) maps the upper half-disk to the lower half-plane,
- (ii) maps the lower half-disk to the upper half-plane,
- (iii) maps the upper half-plane minus upper half-disk to the upper half-plane,
- (iv) maps the lower half-plane minus lower half-disk to the lower half-plane,
- (v) maps the unit circle to the interval $[-1, 1]$.

We now compose the map with the exponential map. Since the exponential map $z \mapsto e^z$ maps the half strip $\{x < 0, 0 < y < \pi\}$ to the upper half-circle, it follows that the map $z \mapsto e^{iz}$ maps the half strip $\{y > 0, 0 < x < \pi\}$ to the upper half-circle. Thus

$$\cos z = \frac{1}{2} (e^{iz} + e^{-iz})$$

maps the half strip $\{y > 0, 0 < x < \pi\}$ to the lower half-plane so that

- (i) $\{0 < x < \pi, y > 0\}$ goes to $[1, \infty)$,
- (ii) $\{0 < x < \pi, y = 0\}$ goes to $[-1, 1]$, and
- (iii) $\{x = \pi, y > 0\}$ goes to $(-\infty, -1]$.