Euler-Lagrange Equations for Many Functions and Variables and High-Order Derivatives

The Case of High-Order Derivatives. The context is to find the extremal for the functional

$$\int_{x=a}^{b} F\left(x, y, y', \cdots, y^{(k)}\right) dx$$

at the function $y = f(x)$ is an extremal compared to other functions nearby, with the jets up to order $k-1$ fixed at the two end-points $x = a$ and $x = b$. In other words, $y^{(j)}(a) = A_j$ and $y^{(j)}(b) = B_j$ for $1 \leq j \leq k-1$.

We assume that we have a family of functions $y = y(x,t)$ parametrized by $t \in (-\varepsilon, \varepsilon)$ so that our assumed solution $y = y(x)$ is $y = y(x,0)$ when the parameter t is 0. Then

$$(\natural)_t \qquad \int_{x=a}^{b} F\left(x, y(x,t), \frac{\partial}{\partial x}y(x,t), \cdots, \frac{\partial^k}{\partial x^k}y(x,t)\right) dx$$

is a function of t and its derivative with respect to t must be equal to 0 at $t = 0$. Differentiating $(\natural)_t$ with respect to t, we get

$$\begin{aligned}
&\frac{d}{dt}\int_{x=a}^{b} F\left(x, y(x,t), \frac{\partial}{\partial x}y(x,t)\right) dx \\
&= \int_{x=a}^{b} \left[F_y\left(x, y(x,t), \frac{\partial}{\partial x}y(x,t), \cdots, \frac{\partial^k}{\partial x^k}y(x,t)\right)\frac{\partial}{\partial t}y(x,t)\right. \\
&\quad + F_{y'}\left(x, y(x,t), \frac{\partial}{\partial x}y(x,t), \cdots, \frac{\partial^k}{\partial x^k}y(x,t)\right)\frac{\partial^2}{\partial x \partial t}y(x,t)dx \\
&\quad + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad \left. + F_{y^{(k)}}\left(x, y(x,t), \frac{\partial}{\partial x}y(x,t), \cdots, \frac{\partial^k}{\partial x^k}y(x,t)\right)\frac{\partial^k}{\partial x^k \partial t}y(x,t)dx\right]
\end{aligned}$$

Setting $t = 0$, we get

$$\begin{aligned}
&\int_{x=a}^{b}\left[F_y\left(x, y, y', \cdots, y^{(k)}\right)\frac{\partial}{\partial t}y(x,0) + F_{y'}\left(x, y, y', \cdots, y^{(k)}\right)\frac{\partial^2}{\partial x \partial t}y(x,0)dx\right. \\
&\quad \left. + \cdots + F_{y^{(k)}}\left(x, y, y', \cdots, y^{(k)}\right)\frac{\partial^k}{\partial x^k \partial t}y(x,0)dx\right] = 0.
\end{aligned}$$

Integrating by parts repeatedly, we obtain

$$\int_{x=a}^{b} \left[F_y\left(x, y, y', \cdots, y^{(k)}\right) - \frac{d}{dx} F_{y'}\left(x, y, y', \cdots, y^{(k)}\right) \right.$$
$$+ \frac{d^2}{dx^2} F_{y''}\left(x, y, y', \cdots, y^{(k)}\right) - + \cdots$$
$$\left. + (-k)^{-1} \frac{d^k}{dx^k} F_{y^{(k)}}\left(x, y, y', \cdots, y^{(k)}\right) \right] \frac{\partial y}{\partial t}(x, 0)\, dx,$$

where all the boundary terms vanish in the process of integration by parts because the fixing of the jets up to order $k-1$ at the two end-points $x = a$ and $x = b$ implies

$$(\sharp) \qquad \frac{d^j}{dx^j} \frac{\partial y}{\partial t}(x, 0) = 0 \quad \text{at} \quad x = a \quad \text{and} \quad x = b \quad \text{for} \quad 1 \leq j \leq k - 1.$$

Since this holds for all choices of $\frac{\partial y}{\partial t}(x, 0)$ for $x \in [a, b]$ as long as the condition $(\sharp)$ is satisfied, we conclude that

$$\sum_{j=0}^{k} (-1)^j \frac{d^j}{dx^j} \left(F_{y^{(j)}}\left(x, y(x), y'(x), \cdots, y^{(k)}(x)\right) \right) \equiv 0$$

on $x \in [a, b]$, which is the Euler-Lagrange equation and is in general a differential equation of order $2k$ if

$$F_{y^{(k)} y^{(k)}}\left(x, y(x), y'(x), \cdots, y^{(k)}(x)\right)$$

is nonzero (which is $(-1)^k$ times the coefficient of $y^{(2k)}(x)$).

The Case of Many Functions. The context is to find the extremal for the functional

$$\int_{x=a}^{b} F\left(x, y_1, y_1', \cdots, y_1^{(k)}, \cdots, y_\ell, y_\ell', \cdots, y_\ell^{(k)}\right) dx$$

at the functions $y_1 = f_1(x), \cdots, y_\ell = f_\ell(x)$ is an extremal compared to other functions nearby, with the jets up to order $k-1$ fixed at the two end-points $x = a$ and $x = b$. In other words, $y_\nu^{(j)}(a) = A_{j,\nu}$ and $y_\nu^{(j)}(b) = B_{j,\nu}$ for $1 \leq j \leq k-1$ and $1 \leq \nu \leq \ell$. We can consider the problem as the problem of variation for one single function $y_\nu = f_\nu(x)$ with all the other functions

$y_\mu = f_\mu(x)$ for $\mu \neq \nu$ fixed. Then we get for each $1 \leq \nu \leq \ell$ an Euler-Lagrange equation

$$\sum_{j=0}^{k}(-1)^j \frac{d^j}{dx^j}\left(F_{y_\nu^{(j)}}\left(x, y_1(x), y_1'(x), \cdots, y_1^{(k)}(x), \cdots, y_\ell(x), y_\ell'(x), \cdots, y_\ell^{(k)}(x)\right)\right) \equiv 0$$

on $x \in [a, b]$.

The Case of Many Independent Variables. Let $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and R be a domain in $\mathbb{R}^n$. The context is to find the extremal for the functional

$$\int_R F\left(\mathbf{x}, \{D^{\mathbf{a}}y_1, \cdots, D^{\mathbf{a}}y_\ell\}_{|a|\leq k}\right) dx_1 \cdots dx_n$$

at the functions $y_1 = f_1(\mathbf{x}), \cdots, y_\ell = f_\ell(\mathbf{x})$ is an extremal compared to other functions on R nearby, with the jets up to order $k-1$ fixed at the boundary ∂R of the given fixed domain R in $\mathbb{R}^n$, where

$$D^{\mathbf{a}} = \frac{\partial^{a_1+\cdots+a_n}}{\partial x_1^{a_1}\cdots \partial x_n^{a_n}}$$

with $\mathbf{a} = (a_1, \cdots, a_n) \in (\mathbb{N} \cup \{0\})^n$ and $|\mathbf{a}| = a_1 + \cdots + a_n$. A completely analogous derivation gives us the following Euler-Lagrange equation in this case for each $1 \leq \nu \leq \ell$

$$\sum_{|\mathbf{a}|\leq k}(-1)^{|\mathbf{a}|}D^{\mathbf{a}}\left(F_{D^{\mathbf{a}}y_\ell}\left(\mathbf{x}, \{D^{\mathbf{a}}y_1, \cdots, D^{\mathbf{a}}y_\ell\}_{|\mathbf{a}|\leq k}\right)\right) \equiv 0$$

for $\mathbf{x} \in R$.

Minimal Surface. We use the equation of a minimal surface as an example of the Euler-Lagrange equation for the case of one function, first-order derivative, and two independent variables. Let R be a bounded domain in $\mathbb{R}^2$ with variables x, y. The problem is to find the Euler-Lagrange equation for a function $z = f(x, y)$ for $(x, y) \in R$ which is a local extremal for the functional

$$\int_R \sqrt{1 + {z_x}^2 + {z_y}^2}\, dxdy$$

of the area of the graph of $z = f(x, y)$ in $\mathbb{R}^3$ over R. The Euler-Lagrange equation is

$$-\frac{\partial}{\partial x}\left(\sqrt{1 + {z_x}^2 + {z_y}^2}\right)_{z_x} - \frac{\partial}{\partial y}\left(\sqrt{1 + {z_x}^2 + {z_y}^2}\right)_{z_y} = 0,$$

which can be rewritten as

$$-\frac{\partial}{\partial x}\frac{z_x}{\sqrt{1+{z_x}^2+{z_y}^2}}-\frac{\partial}{\partial y}\frac{z_y}{\sqrt{1+{z_x}^2+{z_y}^2}}=0,$$

whose expansion is

$$-\frac{z_{xx}}{\sqrt{1+{z_x}^2+{z_y}^2}}+\frac{z_x\left(z_xz_{xx}+z_yz_{xy}\right)}{\left(1+{z_x}^2+{z_y}^2\right)^{\frac{3}{2}}}-\frac{z_{yy}}{\sqrt{1+{z_x}^2+{z_y}^2}}+\frac{z_y\left(z_xz_{xy}+z_yz_{yy}\right)}{\left(1+{z_x}^2+{z_y}^2\right)^{\frac{3}{2}}}=0.$$

After we clear the denominators by multiplying the equation by $\left(1+{z_x}^2+{z_y}^2\right)^{\frac{3}{2}}$, we get

$$z_{xx}\left(1+{z_x}^2+{z_y}^2\right)-z_x\left(z_xz_{xx}+z_yz_{xy}\right)+z_{yy}\left(1+{z_x}^2+{z_y}^2\right)-z_y\left(z_xz_{xy}+z_yz_{yy}\right)=0,$$

which can be simplified to

$$z_{xx}\left(1+{z_y}^2\right)-2z_{xy}z_xz_y+z_{yy}\left(1+{z_x}^2\right)=0.$$

This is the equation for the minimal surface.

Minimal Surface Equation Interpreted as Vanishing of Mean Curvature. We can geometrically interpret the above minimal surface equation in terms of the vanishing of the mean curvature of a surface. First of all let us introduce the notion of curvature for a surface.

Suppose we have a surface S in $\mathbb{R}^3$ given by $\vec{r}=\vec{r}(u,v)$ with parameters u,v. We give S the metric induced from the Euclidean metric of $\mathbb{R}^3$. Then the metric in terms of the coordinates u and v is given by

$$d\vec{r}\cdot d\vec{r}=E\,du^2+2F\,dudv+G\,dv^2,$$

where

$$E=\vec{r}_u\cdot\vec{r}_u,\quad F=\vec{r}_u\cdot\vec{r}_v,\quad G=\vec{r}_v\cdot\vec{r}_v$$

and the subscripts u and v mean partial differentiation with respect to u and v. A metric simply means that the integral of its square root along a curve gives the length of the curve.

Let $\vec{n}$ be the unit normal to the surface S. Choose a point P in the surface S and let C be a curve cut out from S by a plane Π through P containing $\vec{n}(P)$. Let $u=u(t)$, $v=v(t)$ be the equations defining C with the parameter t equal to the arc-length of C. We now compute the curvature of the curve C.

Let $\vec{r}'$ and $\vec{r}''$ denote respectively the first and second order derivatives of $\vec{r}$ with respect to t. Since $\vec{r}'$ is a unit vector, the curvature of C is given by the length of $\vec{r}''$. Since C is a plane curve on the same plane as $\vec{n}(P)$, the vector $\vec{r}''(P)$ is parallel to $\vec{n}(P)$. Hence the curvature κ of C at P is given by $\vec{r}'' \cdot \vec{n}$ which by the chain rule is equal to

$$(\vec{r}_{uu} \cdot \vec{n}){u'}^2 + 2(\vec{r}_{uv} \cdot \vec{n})u'v' + (\vec{r}_{vv} \cdot \vec{n}){v'}^2,$$

where the primes for u and v mean differentiation with respect to t. Let $D = \vec{n} \cdot \vec{r}_{uu}$, $D' = \vec{n} \cdot \vec{r}_{uv}$, and $D'' = \vec{n} \cdot \vec{r}_{vv}$. Since t is the arc-length of C, we can write

$$\kappa = \frac{D\ du^2 + 2D'dudv + D''dv^2}{E\ du^2 + 2F\ dudv + F\ dv^2}.$$

The expression makes κ the quotient of two quadratic forms whose variables are the components of the tangent vector at P which belongs to Π. As we change this tangent vector or equivalently as we change Π, the value of κ changes and we have two extremal values. The product of these two extremal values is known as the *Gaussian curvature*. The sum of these two extremal values is known as the *mean curvature*. We let $\xi = \frac{du}{dt}$ and $\eta = \frac{dv}{dt}$. We have

$$\kappa(\xi, \eta)\left(E\,\xi^2 + 2F\,\xi\eta + G\,\eta^2\right) = D\,\xi^2 + 2D'\xi\eta + D''\eta^2.$$

By differentiating with respect to ξ and η, we conclude that the extremal values of κ satisfy the two equations

$$\begin{aligned}(D - \kappa E)\xi + (D' - \kappa F)\eta &= 0\\ (D' - \kappa F)\xi + (D'' - \kappa G)\eta &= 0.\end{aligned}$$

The vanishing of the determinant of the coefficients of ξ and η yields the following quadratic equation in κ

$$\begin{vmatrix} D - \kappa E & D' - \kappa F \\ D' - \kappa F & D'' - \kappa G \end{vmatrix} = 0,$$

whose expansion is

$$(D - \kappa E)\,(D'' - \kappa G) - (D' - \kappa F)^2 = 0,$$

which can be simplified to

$$\left(EG - F^2\right)\kappa^2 + \left(2D'F - DG - ED''\right)\kappa + \left(DD'' - {D'}^2\right) = 0.$$

The product of the two roots of κ in this equation can be read off from the coefficients of the equation. So we conclude the Gaussian curvature is given by
$$\frac{DD'' - D'^2}{EG - F^2}.$$
Another way to see it is that the Gaussian curvature is the product of the eigenvalues of the matrix
$$\begin{pmatrix} D & D' \\ D' & D'' \end{pmatrix}$$
with respect to the matrix
$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}$$
and so is equal to the quotient of the determinants of the two matrices.

Likewise the sum of the two roots of κ in this equation can also be read off from the coefficients of the equation. So we conclude the mean curvature is given by
$$\frac{DG + ED'' - 2D'F}{EG - F^2}.$$
The vanishing of the mean curvature is the same as
$$DG + ED'' - 2D'F = 0.$$
For our case when the surface S is given as the graph of $z = z(x, y)$, the two parameters are $u = x$ and $v = y$ and we have the parametric representation
$$\vec{r}(x, y) = (x, y, z(x, y)).$$
Hence we have the following expressions for E, F, G in terms of $z(x, y)$ and its derivatives up to order 2.
$$\begin{gathered} E = \vec{r}_x \cdot \vec{r}_x = (1, 0, z_x) \cdot (1, 0, z_x) = 1 + {z_x}^2, \\ F = \vec{r}_x \cdot \vec{r}_y = (1, 0, z_x) \cdot (0, 1, z_y) = z_x z_y, \\ G = \vec{r}_y \cdot \vec{r}_y = (0, 1, z_y) \cdot (0, 1, z_y) = 1 + {z_y}^2, \end{gathered}$$
For the computation of D, D', D'' in terms of $z(x, y)$ and its derivatives up to order 2, we first compute $\vec{n}$ as the unit vector in the direction of $\vec{r}_x \times \vec{v}_y$. Now we have
$$\vec{r}_x \times \vec{r}_y = (1, 0, z_x) \times (0, 1, z_y) = (-z_x, -z_y, 1)$$

from which we get

$$\vec{n} = \frac{1}{\sqrt{1 + {z_x}^2 + {z_y}^2}} \left(-z_x, -z_y, 1\right).$$

Hence we have the following expressions for D, D', D'' in terms of $z(x, y)$ and its derivatives up to order 2.

$$D = \vec{n} \cdot \vec{r}_{xx} = \frac{(-z_x, -z_y, 1) \cdot (0, 0, z_{xx})}{\sqrt{1 + {z_x}^2 + {z_y}^2}} = \frac{z_{xx}}{\sqrt{1 + {z_x}^2 + {z_y}^2}},$$
$$D' = \vec{n} \cdot \vec{r}_{xy} = \frac{(-z_x, -z_y, 1) \cdot (0, 0, z_{xy})}{\sqrt{1 + {z_x}^2 + {z_y}^2}} = \frac{z_{xy}}{\sqrt{1 + {z_x}^2 + {z_y}^2}},$$
$$D'' = \vec{n} \cdot \vec{r}_{yy} = \frac{(-z_x, -z_y, 1) \cdot (0, 0, z_{yy})}{\sqrt{1 + {z_x}^2 + {z_y}^2}} = \frac{z_{xx}}{\sqrt{1 + {z_x}^2 + {z_y}^2}},$$

The condition of the vanishing of the mean curvature

$$DG + ED'' - 2D'F = 0$$

becomes

$$z_{xx} \left(1 + {z_y}^2\right) - 2 z_{xy} z_x z_y + z_{yy} \left(1 + {z_x}^2\right)^2 = 0,$$

which is precisely the Euler-Lagrange equation we derived earlier for minimal surface.