Euler-Lagrange Equations for One Function of One Variable With Fixed End-Points and One Order of Differentiation

Context. Fix $-\infty < a < b < \infty$ and $A, B \in \mathbb{R}$. Fix a continuously differentiable function $F = F(x, y, y')$ of three independent variables x, y, y'. We consider the set of all twice continuously differentiable functions $y = f(x)$ on $[a, b]$ with $f(a) = A$ and $f(b) = B$. We investigate the problem of finding a function $y = f(x)$ such that the value of

$$(\natural) \qquad \int_{x=a}^{b} F(x, y, y')\, dx$$

at the function $y = f(x)$ is an extremal compared to other functions nearby (in the sense that the derivatives up to second order of the difference are small).

Logic to Derive Necessary Condition. The logic is to assume that a solution is known and then derive necessary conditions which must be satisfied by the assumed solution. Suppose $y = y(x)$ is a solution. We are going to derive the necessary condition by comparing the value of the functional $(\natural)$ at $y = y(x)$ to the value of the functional $(\natural)$ at some other functions $y = y(x, t)$ near $y = y(x)$.

We assume that we have a family of functions $y = y(x, t)$ parametrized by $t \in (-\varepsilon, \varepsilon)$ so that our assumed solution $y = y(x)$ is $y = y(x, 0)$ when the parameter t is 0. Then

$$(\natural)_t \qquad \int_{x=a}^{b} F\left(x, y(x,t), \frac{\partial}{\partial x} y(x,t)\right) dx$$

is a function of t and its derivative with respect to t must be equal to 0 at $t = 0$. Differentiating $(\natural)_t$ with respect to t, we get

$$\begin{aligned} &\frac{d}{dt} \int_{x=a}^{b} F\left(x, y(x,t), \frac{\partial}{\partial x} y(x,t)\right) dx \\ = &\int_{x=a}^{b} \left(F_y\left(x, y(x,t), \frac{\partial}{\partial x} y(x,t)\right) \frac{\partial}{\partial t} y(x,t) \right. \\ &\left. + F_{y'}\left(x, y(x,t), \frac{\partial}{\partial x} y(x,t)\right) \frac{\partial^2}{\partial x \partial t} y(x,t) \right) dx. \end{aligned}$$

Setting $t = 0$, we get

$$\int_{x=a}^{b} \left(F_y\left(x, y(x), y'(x)\right) \frac{\partial y}{\partial t}(x,0) + F_{y'}\left(x, y(x), y'(x)\right) \frac{\partial^2 y}{\partial x \partial t}(x,0) \right) dx = 0.$$

Integrating by parts, we obtain

$$\int_{x=a}^{b} \left(F_y\left(x, y(x), y'(x)\right) - \frac{d}{dx} F_{y'}\left(x, y(x), y'(x)\right) \right) \frac{\partial y}{\partial t}(x,0)\, dx$$
$$+ F_{y'}\left(x, y(x), y'(x)\right) \frac{\partial y}{\partial t}(x,0) \bigg|_{x=a}^{x=b} = 0.$$

By assumption, all functions used in the comparison assume value A at $x = a$ and assume value B at $x = b$. Hence we must have

$$(\sharp) \qquad \frac{\partial y}{\partial t}(a,0) = \frac{\partial y}{\partial t}(b,0) = 0$$

and we conclude that

$$\int_{x=a}^{b} \left(F_y\left(x, y(x), y'(x)\right) - \frac{d}{dx} F_{y'}\left(x, y(x), y'(x)\right) \right) \frac{\partial y}{\partial t}(x,0)\, dx = 0$$

for all choices of $\frac{\partial y}{\partial t}(x,0)$ for $x \in [a,b]$ as long as the condition $(\sharp)$ is satisfied.

$$F_y\left(x, y(x), y'(x)\right) - \frac{d}{dx} F_{y'}\left(x, y(x), y'(x)\right) \equiv 0$$

on $x \in [a,b]$, which is the Euler-Lagrange equation.

Right-Hand Side of Euler-Lagrange Equation as Gradient of Functional. The expression

$$\int_{x=a}^{b} \left(F_y\left(x, y(x), y'(x)\right) - \frac{d}{dx} F_{y'}\left(x, y(x), y'(x)\right) \right) \frac{\partial y}{\partial t}(x,0)\, dx$$

derived above is the derivative of the functional $(\natural)$ along the vector

$$\frac{\partial y}{\partial t}(x,0)$$

in the space of functions y of x with $y(a) = A$ and $y(b) = B$. In the case of a finite number of variables, the derivative along a vector is equal to the inner product of the gradient and the vector. Now the inner product is replaced by the integral of the product of two functions. In that sense we can regard the right-hand side

$$F_y\left(x, y(x), y'(x)\right) - \frac{d}{dx} F_{y'}\left(x, y(x), y'(x)\right)$$

of the Euler-Lagrange equation as the gradient of the functional (♮).

First Integral as Analog of Conservation of Energy When the Integrand in the Functional Is Independent of the Independent Variable. When the integrand $F\left(x, y, y'\right)$ is independent of the independent variable x, we have the first integral $F_{y'} =$ constant for the Euler-Lagrange equation which is the analog of the conservation of energy. We will explain why this first integral is the analog of the conservation of energy when we later discuss the Legendre transformation and Hamiltonian differential equations. Now we verify here that indeed what we write down as a first integral is a first integral. The verification of $F - y'F_{y'} =$ constant is as follows.

$$\begin{aligned} \frac{d}{dx}\left(F - y'F_{y'}\right) &= F_y y' + F_{y'} y'' - \left(y'' F_{y'} + y' \frac{d}{dx} F_{y'}\right) \\ &= y' \left(F_y - \frac{d}{dx} F_{y'}\right) = 0. \end{aligned}$$

Discrete Analog. We discuss the discrete analog of the Euler-Lagrange equation for the case when $F\left(x, y, y'\right)$ to illustrate the relationship between the calculus of variations and the calculus of a finite number of variables.

We go to the discrete situation in which $[a, b]$ is replaced by $\{0, 1, \cdots, n, n+1\}$ and the finite number of variables are $y_0, y_1, \cdots, y_n, y_{n+1}$ which are the values of the function y at the points of the finite set $\{0, 1, \cdots, n, n+1\}$. The functional

$$\int_{x=a}^{b} F\left(y, y'\right) \, dx$$

is now replaced by

$$(\dagger) \qquad \begin{aligned} &F\left(y_0, y_1 - y_0\right) + F\left(y_1, y_2 - y_1\right) + F\left(y_2, y_3 - y_2\right) \\ &\quad + \cdots + F\left(y_{n-1}, y_n - y_{n-1}\right) + F\left(y_n, y_{n+1} - y_n\right). \end{aligned}$$

Integration by parts is replaced by Abel's summation by parts. If $s_j = a_1 + a_2 + \cdots + a_j$, then

$$\begin{aligned} & a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \\ & = b_1 s_1 + b_2 (s_2 - s_1) + \cdots + b_n (s_n - s_{n-1}) \\ & = s_1 (b_1 - b_2) + s_2 (b_2 - b_3) + \cdots + s_{n-1} (b_{n-1} - b_n) + s_n b_n . \end{aligned}$$

Differentiating (†) yields

$$(\ddagger) \qquad \begin{aligned} & F_1 (y_0, y_1 - y_0) y_0{}' + F_2 (y_0, y_1 - y_0) (y_0{}' - y_0{}') \\ & + F_1 (y_1, y_2 - y_1) y_1{}' + F_2 (y_1, y_2 - y_1) (y_2{}' - y_1{}') \\ & + F_1 (y_2, y_3 - y_2) y_2{}' + F_2 (y_2, y_3 - y_2) (y_3{}' - y_2{}') \\ & + \cdots\cdots\cdots\cdots \\ & + F_1 (y_{n-1}, y_n - y_{n-1}) y_{n-1}{}' + F_2 (y_{n-1}, y_n - y_{n-1}) (y_n{}' - y_{n-1}{}') \\ & + F_1 (y_n, y_{n+1} - y_n) y_n{}' + F_2 (y_n, y_{n+1} - y_n) (y_{n+1}{}' - y_n{}') , \end{aligned}$$

where F_1 means partial derivative with respect to its first variable and F_2 means partial derivative with respect to its second variable. We now apply Abel's summation by parts to

$$\sum_{\nu=0}^{n} F_2 (y_\nu , \, y_{\nu+1} - y_\nu) (y_{\nu+1}{}' - y_\nu{}')$$

to get

$$\begin{aligned} & \sum_{\nu=0}^{n} F_2 (y_\nu , \, y_{\nu+1} - y_\nu) (y_{\nu+1}{}' - y_\nu{}') \\ & = -F_2 (y_0 , \, y_1 - y_0) y_0{}' + F_2 (y_n , \, y_{n+1} - y_n) y_{n+1}{}' \\ & - \sum_{\nu=0}^{n-1} \left(F_2 (y_{\nu+1} , \, y_{\nu+2} - y_{\nu+1}) - F_2 (y_\nu , \, y_{\nu+1} - y_\nu) \right) y_{\nu+1}{}' . \end{aligned}$$

Thus we can rewrite (††) as

$$\begin{aligned} & \sum_{\nu=0}^{n} F_1 (y_\nu , \, y_{\nu+1} - y_\nu) y_\nu{}' \\ & - F_2 (y_0 , \, y_1 - y_0) y_0{}' + F_2 (y_n , \, y_{n+1} - y_n) y_{n+1}{}' \end{aligned}$$

$$- \sum_{\nu=0}^{n-1} \left(F_2\left(y_{\nu+1},\, y_{\nu+2} - y_{\nu+1}\right) - F_2\left(y_\nu,\, y_{\nu+1} - y_\nu\right)\right) {y_{\nu+1}}'$$

$$= \left(F_1\left(y_0,\, y_1 - y_0\right) - F_2\left(y_0,\, y_1 - y_0\right)\right) {y_0}' + F_2\left(y_n,\, y_{n+1} - y_n\right) {y_{n+1}}'$$

$$+ \sum_{\nu=0}^{n-1} \left(F_1\left(y_\nu,\, y_{\nu+1} - y_\nu\right) - \left(F_2\left(y_{\nu+1},\, y_{\nu+2} - y_{\nu+1}\right) - F_2\left(y_\nu,\, y_{\nu+1} - y_\nu\right)\right)\right) {y_{\nu+1}}'.$$

When the two end-points at 0 and at $n+1$ are fixed (that is, ${y_0}' = 0$ and ${y_{n+1}}' = 0$), we end up with final expression

$$\sum_{\nu=0}^{n-1} \left(F_1\left(y_\nu,\, y_{\nu+1} - y_\nu\right) - \left(F_2\left(y_{\nu+1},\, y_{\nu+2} - y_{\nu+1}\right) - F_2\left(y_\nu,\, y_{\nu+1} - y_\nu\right)\right)\right) {y_{\nu+1}}',$$

which means that the gradient of (†) is

$$\left\{F_1\left(y_\nu,\, y_{\nu+1} - y_\nu\right) - \left(F_2\left(y_{\nu+1},\, y_{\nu+2} - y_{\nu+1}\right) - F_2\left(y_\nu,\, y_{\nu+1} - y_\nu\right)\right)\right\}_{\nu=0}^{n-1}.$$